Chapter 5 roadmap:

- Introduction to relations (Wednesday)
- Properties of relations (**Today** and Wednesday)
- Closures (Friday)
- Partial order relations (next week Monday)

"Today" (Monday and Wednesday):

- Review of definitions from last time
- Revisit proofs from alst time
- Hints on homework problems
- Properties of relations
  - Reflexivity
  - Symmetry
  - Transitivity
- Proofs
- More proofs

Coming up:

Due **Monday***:*
Take quiz on Section 5.(1&2), read Section 5.3

Due **Tuesday***:*
Do Exercises 5.1.5 and 5.2.(7, 8, 10, 11, 12, 13, 13b, 14)
See Canvas for hints/explanations.

Due **Wednesday***:*
Take quiz on Section 5.3

Due **Friday***:*
Do Exercises 5.3.(2, 3, 4, 21, 23, 24, 34, 36, 37)
Read Section 5.4
Take quiz on Section 5.4

Consider the set of students $\{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}$. Suppose they all sit in the front row, with this seating arrangement:

Dave	Alice	Carol	Bob

Consider the relation *sitsNextTo* on this set. Determine which of the following are true.

$\text{Carol} \in \textit{sitsNextTo}$

$(\text{Dave}, \text{Alice}) \in \textit{sitsNextTo}$

$(\text{Dave}, \text{Bob}) \in \textit{sitsNextTo}$

$(\text{Alice}, \text{Carol}) = \textit{sitsNextTo}$

$\textit{sitsNextTo} = \{\text{Dave}, \text{Alice}, \text{Carol}, \text{Bob}\}$

$\textit{sitsNextTo} = \{(\text{Dave}, \text{Alice}), (\text{Alice}, \text{Carol}), (\text{Carol}, \text{Bob})\}$.

$\textit{sitsNextTo} =$
$\{(\text{Alice}, \text{Carol}), (\text{Alice}, \text{Dave}), (\text{Bob}, \text{Carol}), (\text{Carol}, \text{Alice}), (\text{Carol}, \text{Bob}), (\text{Dave}, \text{Alice})\}$

A **relation** from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A **relation** on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The **image** of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The **image** of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The **inverse** of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The **composition** of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The **identity** relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	$=$

Theorem 5.1 If $a, b \in \mathbb{N}$ and $a|b$, then $\mathcal{I}_|(b) \subseteq \mathcal{I}_|(a)$.

Proof. *Suppose* $a, b \in \mathbb{N}$ *and* $a|b$. *By definition of divides, there exists* $i \in \mathbb{N}$ *such that* $a \cdot i = b$.

Suppose further that $c \in \mathcal{I}_|(b)$. *By definition of image,* $b|c$. *By definition of divides, there exists* $j \in \mathbb{N}$ *such that* $b \cdot j = c$.

By substitution, $a \cdot i \cdot j = c$, *and so* $a|c$ *by definition of divides. By definition of image,* $c \in \mathcal{I}_|(a)$, *and by definition of subset,* $\mathcal{I}_|(b) \subseteq \mathcal{I}_|(a)$. $\square$

Theorem 5.2 If R is a relation on a set A, $a \in A$, and $\mathcal{I}_R(a) \neq \emptyset$, then $a \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(a))$.

Proof. *Suppose R is a relation on A, $a \in A$, and $\mathcal{I}_R(a) \neq \emptyset$.*

Let $b \in \mathcal{I}_R(a)$. By definition of image, $(a, b) \in R$. By definition of inverse, $(b, a) \in R^{-1}$. By definition of image (extended for sets), $a \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(a))$. $\square$

Ex 5.2.7. Prove that if R is a relation on a set A and $(a, b) \in R$, then $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$.

Ex 5.2.8. Suppose R is a relation from a set X to a set Y and $A \subseteq X$. Is the following true?
$\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$.
Prove or give a counterexample for each.

Attempted proof. *Suppose $x \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(A))$. [We want $x \in A$.]*

By definition of image, there exists $y \in \mathcal{I}_R(A)$ such that $(y, x) \in R^{-1}$.

[From $y \in \mathcal{I}_R(A)$]
By definition of image, there exists $a \in A$ such that $(a, y) \in R$.

[From $(y, x) \in R^{-1}$]
By definition of relation inverse, $(x, y) \in R$

[We know $a \in A$, and both $(a, y) \in R$ and $(x, y) \in R$. Could it be that $a = x$? Doesn't seem to be a way to prove that... I seem stuck]

Counterexample. *Let $X = \{x, a\}$, $A = \{a\}$, and $Y = \{y\}$.*
Let $R = \{(x, y), (a, y)\}$.
Then $R^{-1} = \{(y, x), (y, a)\}$, $\mathcal{I}_R(A) = \{y\}$, and $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) = \{x, a\}$
In this example, $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \not\subseteq A$.

Ex 5.2.9. Prove that if R is a relation from A to B, then $i_B \circ R = R$.

Proof. *First suppose* $(x, y) \in i_B \circ R$. *By definition of composition, there exists* $b \in B$ *such that* $(x, b) \in R$ *and* $(b, y) \in i_B$.

By definition of the identity relation, $b = y$. *By substitution,* $(x, y) \in R$. *Hence* $i_B \circ R \subseteq R$ *by definition of subset.*

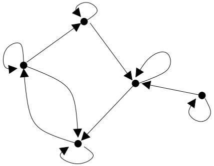
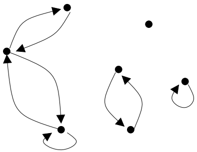
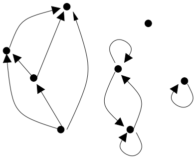
Next suppose $(x, y) \in R$. *By how* R *is defined, we know* $x \in A$ *and* $y \in B$.

By definition of the identity relation, $(y, y) \in i_B$. *By definition of composition,* $(x, y) \in i_B \circ R$. *Hence* $R \subseteq i_B \circ R$.

Therefore, by definition of set equality, $i_B \circ R = R$. $\square$

Ex 5.2.10. $(R^{-1})^{-1} = R$.

Ex 5.2.11. If R is a relation from A to B, is $R^{-1} \circ R = i_A$? Prove or give a counterexample.

	Reflexivity	**Symmetry**	**Transitivity**
Informal	Everything is related to itself	All pairs are mutual	Anything reachable by two hops is reachable by one hop
Formal	$\forall\ x \in X,\ (x, x) \in R$	$\forall\ x, y\ \in\ X,\ (x, y)\ \in\ R\ \rightarrow (y, x) \in R$ OR $\forall (x, y) \in R,\ (y, x) \in R$	$\forall x, y, z \in X,$ $(x, y), (y, z) \in R \rightarrow (x, z) \in R$ OR $\forall (x, y), (y, z) \in R, (x, z) \in R$
Visual			
Examples	$\subseteq$, $\leq$, $\geq$, $\equiv$, i, isAquaintedWith, waterVerticallyAligned	$\equiv$, isOppositeOf, isOnSameRiver, isAquaintedWith	$<$, $\leq$, $>$, $\geq$, $\subseteq$, isTallerThan, isAncestorOf, isWestOf

	Reflexivity	**Symmetry**	**Transitivity**
Formal	$\forall\ x \in X$, $(x, x) \in R$	$\forall\ x, y \in X$, $(x, y) \in R \to (y, x) \in R$ OR $\forall\ (x, y) \in R$, $(y, x) \in R$	$\forall\ x, y, z \in X$, $(x, y), (y, z) \in R \to (x, z) \in R$ OR $\forall\ (x, y), (y, z) \in R, (x, z) \in R$
Analytical use	Suppose R is reflexive and $a \in X$. Then $(a, a) \in R$.	Suppose R is symmetric $[a, b \in X]$ and $(a, b) \in R$. Then $(b, a) \in R$	Suppose R is transitive $[a, b, c \in X]$ and $(a, b), (b, c) \in R$. Then $(a, c) \in R$.
Synthetic use	Suppose $a \in X$. ... $(a, a) \in R$. Hence R is reflexive.	Suppose $(a, b) \in R$. ... $(b, a) \in R$. Hence R is symmetric.	Suppose $(a, b), (b, c) \in R$. ... $(a, c) \in R$. Hence R is transitive.

Theorem 5.5. $\mid$ (divides) is reflexive.

Exercise 5.3.1. $|$ (divides) is not symmetric.

Theorem 5.6. $R \cap R^{-1}$ is symmetric.

Theorem 5.7. $\mid$ is transitive.

Exercise 5.3.19. $R^{-1} \circ R$ is reflexive. *(False)*

Exercise 5.3.20. If R and S are both reflexive, then $R \cap S$ is reflexive.

Exercise 5.3.22. If R and S are both symmetric, then $(S \circ R) \cup (R \circ S)$ is symmetric.

Based on Exercise 5.3.32. If R is transitive, then $R \circ R \subseteq R$.

Exercise 5.3.26. If R is transitive, $\mathcal{I}_R(\mathcal{I}_R(A)) \subseteq \mathcal{I}_R(A)$.

Exercise 5.3.31. If R is reflexive and

$$(\text{for all } a, b, c \in A, \text{ if } (a, b) \in R \text{ and } (b, c) \in R \text{ then } (c, a) \in R),$$

then R is an equivalence relation.

For next time (Friday, Nov 1):

Due **Friday***:*

Do Exercises 5.3.(2, 3, 4, 21, 23, 24, 34, 36, 37)

Read Section 5.4

Take quiz on Section 5.4