Counting and combinatorics unit

- Cardinality and countability (Wednesday)
- Combinations and permutations (**Today**)
- The pigeonhole principle (next week Monday)

Today:

- The multiplication rule (Cartesian products)
- The “power rule” (subsets/powersets)
- Combinations
- Permutations
- (Time permitting) Balls and urns

- A game involves two six-sided dice. To distinguish the two pieces, assume one is white and the other red. How many distinct roles are possible?
- A person owns two pairs of jeans and eight T-shirts. How many distinct outfits can be made from these?
- An ice cream parlor sells sundaes with one scoop of ice cream, one squirt of sauce, and one spoonful of a topping. The ice cream flavors are vanilla, chocolate, and mint; the sauces are chocolate, caramel, strawberry, and butterscotch; the toppings are walnuts, sprinkles, chocolate chips, M&Ms, and cherries. How many different sundaes can be ordered?
- A table consists in 15 columns and 8 rows. How many positions are in the table?
- A flowering plant species is being studied. Some individuals have red flowers, others have pink flowers. Some individuals have straight leaves, others curled, and others wavey. Various individuals produce seed pods with two, three, or four seeds, but on any individual all pods have the same number of seeds. How many ways can these characteristics can be combined?

- A game consists in collecting items represented on cards, with each card uniquely representing one item. There are 30 cards, and there is no limit on the number of cards that a player can acquire. How many possible hands are there?
- A person has 12 necklaces. How many ways can those necklaces be combined to wear together?
- A rival ice cream parlor serves only vanilla ice cream, but has the same selection of sauces and toppings. Moreover, this parlor doesn't differentiate between sauces and toppings, and allows any number of different sauces and toppings. How many different sundaes can be ordered?
- Subsets of a set of size 8 are represented using Boolean sequences of length 8. How many such Boolean sequences are there?
- In another flowering plan species being studied, there are 16 characteristics, seemingly unrelated, that individual plants either have or don't have—for example, curled or straight leaves, flaky or wet pollen, smooth or hairy stems. How many ways can these characteristics be combined?

The powerset of the set $\{\bullet, \heartsuit, \bigstar, \spadesuit\}$, arranged by size of the constituent sets:

Size of the sets	Sets themselves	Number of sets of that size
0	$\varnothing$	1
1	$\{\bullet\}, \{\heartsuit\}, \{\bigstar\}, \{\spadesuit\}$	4
2	$\{\bullet, \heartsuit\}, \{\bullet, \bigstar\}, \{\bullet, \spadesuit\}, \{\heartsuit, \bigstar\}, \{\heartsuit, \spadesuit\}, \{\bigstar, \spadesuit\}$	6
3	$\{\bullet, \heartsuit, \bigstar\}, \{\bullet, \heartsuit, \spadesuit\}, \{\bullet, \bigstar, \spadesuit\}, \{\spadesuit, \heartsuit, \bigstar\}$	4
4	$\{\bullet, \heartsuit, \bigstar, \spadesuit\}$	1

The number of **combinations** of size r that can be drawn from set n is

$$\binom{n}{r} = C_r^n = \frac{n!}{(n-r)!r!}$$

$$\begin{aligned}
16 &= 1 + 4 + 6 + 4 + 1 \\
&= \frac{24}{24} + \frac{24}{6} + \frac{24}{4} + \frac{24}{6} + \frac{24}{24} \\
&= \frac{24}{24\cdot 1} + \frac{25}{6\cdot 1} + \frac{24}{2\cdot 2} + \frac{24}{1\cdot 6} + \frac{24}{1\cdot 24} \\
&= \frac{24}{4!\cdot 0!} + \frac{24}{3!\cdot 1!} + \frac{24}{2!\cdot 2!} + \frac{24}{1!\cdot 3!} + \frac{24}{0!\cdot 4!}
\end{aligned}$$

The permutations of size 3 that can be drawn from the set {●, ♡, ★, ♠}:

{●, ♡, ★} [●, ♡, ★] [●, ★, ♡] [♡, ●, ★]
[♡, ★, ●] [★, ●, ♡] [★, ♡, ●]

{●, ♡, ♠} [●, ♡, ♠] [●, ♠, ♡] [♡, ●, ♠]
[♡, ♠, ●] [♠, ♡, ●] [♠, ●, ♡]

{●, ★, ♠} [●, ★, ♠] [●, ♠, ★,] [★, ●, ♠]
[★, ♠, ●,] [♠, ●, ★,] [♠, ★, ●,]

{♠, ♡, ★} [♠, ♡, ★] [♠, ★, ♡] [♡, ♠, ★]
[♡, ★♠] [★, ♠, ♡] [★, ♡♠]

The number of **permutations** of an entire set of size n is $n!$; the number of permutations of size r that can be drawn from a set of size n is

$$P_r^n = \frac{n!}{(n-r)!}$$

What is a permutation? Four perspectives:

- A sequence: $[\spadesuit, \bullet, \bigstar, \heartsuit]$
- A total order: $\spadesuit \preceq \bullet \preceq \bigstar \preceq \heartsuit$
- A one-to-one correspondence:

0	$\spadesuit$
1	$\bullet$
2	$\bigstar$
3	$\heartsuit$

- A nested tuple: $[\spadesuit, \bullet, \bigstar, \heartsuit] = (\spadesuit, [\bullet, \bigstar, \heartsuit]) = (\spadesuit, (\bullet, (\bigstar, (\heartsuit))))$

Theorem 1

Let X be a finite set and $n, r \in \mathbb{W}$. If $n = |X|$ and $r \leq n$, then the number of permutations of size r drawn from X is

$$P_r^n = \frac{n!}{(n-r)!}$$

Theorem 2 (Theorem 1 restated)

For all $n \in \mathbb{W}$, for all $r \in \mathbb{W}$, for all finite sets X, if $r \leq n$ and $|X| = n$, then the number of permutations of size r drawn from X is

$$P_r^n = \frac{n!}{(n-r)!}$$

Theorem 3

For all $n \in \mathbb{W}$, *for all* $r \in \mathbb{W}$, *for all finite sets* X, *if* $r \leq n$ *and* $|X| = n$, *then the number of combinations of size* r *drawn from* X *is*

$$\binom{n}{r} = C_r^n = \frac{n!}{(n-r)!r!}$$

- Suppose there are n balls in a single urn, and assume that all the balls are easily distinguishable from each other. You grab balls from the urn one at a time, and at each draw you take note of the ball and *put it back in the urn*. These draws are ordered, but you could pull out the same ball multiple times. This is called **drawing with replacement**. How many different sequences of r draws are possible?
- Next imagine the same situation—n distinguishable balls in a single urns, drawn one a time, r times—but this time assume that we don't put the ball back in the urn after we draw it. Thus each ball may occur at most once in the sequence. How many distinct sequences are possible?
- Finally, imagine the same situation as before, but instead of pulling r balls one at a time, we grab all r balls in one draw. How many ways to draw the balls are possible?

For next time:

Do the exercises on the back of the handout (to turn in at class time).