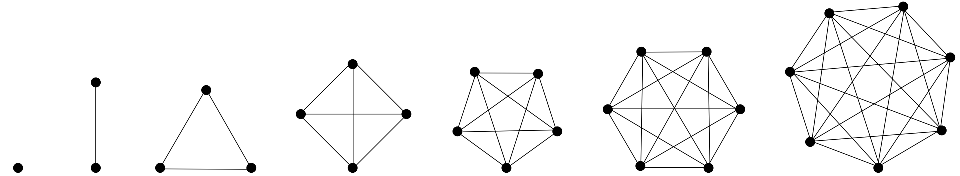
Graph unit

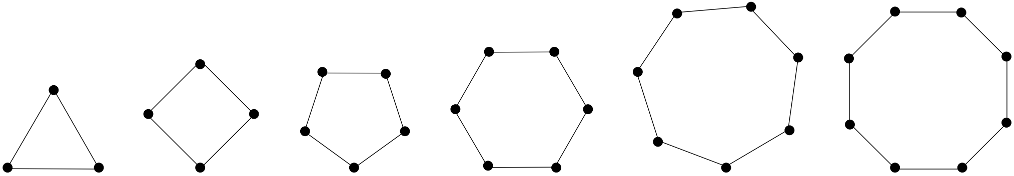
- A little more Bayesian inference; begin graph theory (last week Friday)
- Graph proofs (Monday)
- Graph isomorphisms (Wednesday)
- Varieties of graphs (**today**)
- Graphs as models of information (next week Monday)

Today:

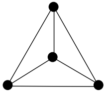
- Complete graphs, cycles, wheels, n-cubes
- Bipartite graphs, planar graphs
- Cliques, strongly connected components, articulation points, vertex covers
- Trees
- Applications: Game theory, flow networks



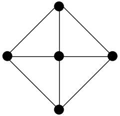
K_1 K_2 K_3 K_4 K_5 K_6 K_7



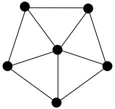
C_3 C_4 C_5 C_6 C_7 C_8



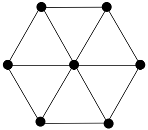
W_3



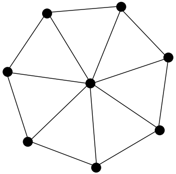
W_4



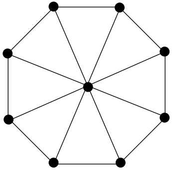
W_5



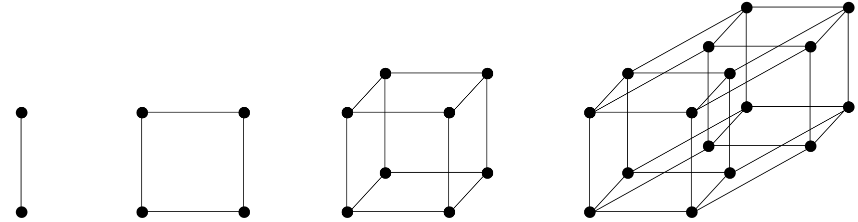
W_6



W_7



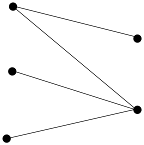
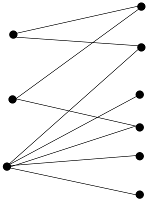
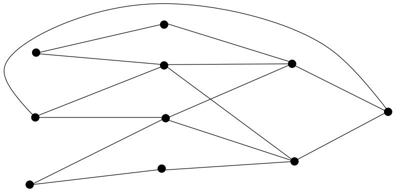
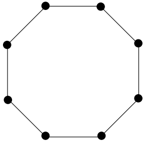
W_8

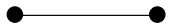


Q_1 Q_2 Q_3 Q_4

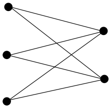
Let $Q_1 = (\{v_1, v_2\}, \{(v_1, v_2)\})$. For $n \in \mathbb{N}$, $n \geq 1$, let $Q_n = (V \cup W, E \cup F \cup R)$ where

- $V \cap W = \varnothing$ and E, F, R are pairwise disjoint (so V and W make a partition of the vertices and E, F, and R make a partition of the edges).
- (V, E) and (W, F) each make a subgraph isomorphic to Q_{n-1}.
- There exists an isomorphism $(\phi : V \to W, \psi : E \to F)$ such that $R = \{(v, \phi(v)) \mid v \in V\}$.

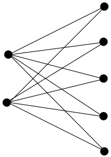







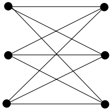
$K_{1,1}$



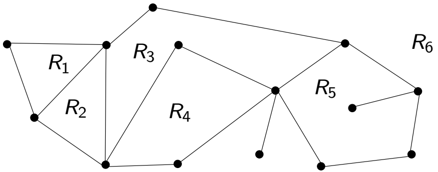
$K_{3,2}$



$K_{2,5}$

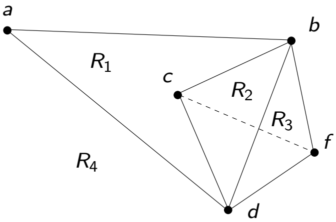


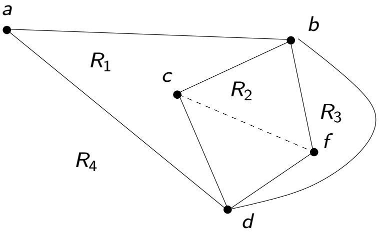
$K_{3,3}$



A graph is **planar** if

- it has exactly one edge. in which case it also has one region; or
- if for every edge $e \in E$ with endpoints v and w, we have that the graph with e taken away $(G_1 = (V, E - \{e\}))$ is itself planar and either v and w are disconnected or there exists for G_1 a certificate set of cycles $R_1, R_2, \ldots R_n$ such that v and w are both on some cycle R_i.

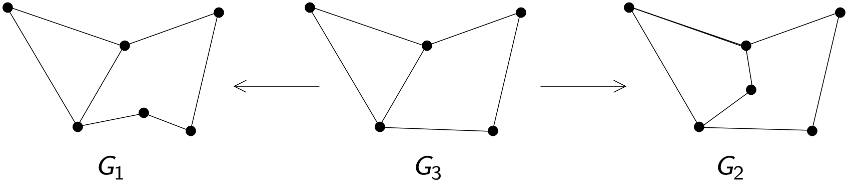




Theorem (Euler's Formula)

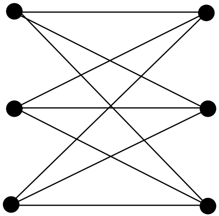
If $G = (V, E)$ *is a connected, simple, planar graph with certificate* $\{R_1, R_2, \ldots, R_n\}$*, then* $n = |E| - |V| + 2$.

Two graphs G_1, G_2 are **homeomorphic** if they can each be derived from a third graph G_3 by repeatedly picking an edge from G_3 and breaking it with a new, intervening vertex.

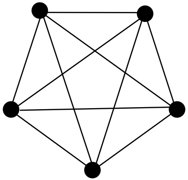


Theorem (Kuratowski's Theorem)

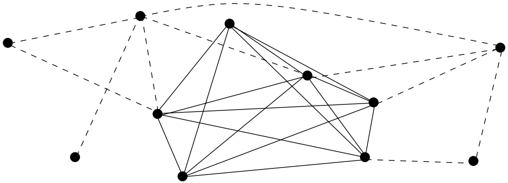
A graph G is not planar iff there exists a subgraph of G that is homeomorphic to $K_{3,3}$ or K_5.



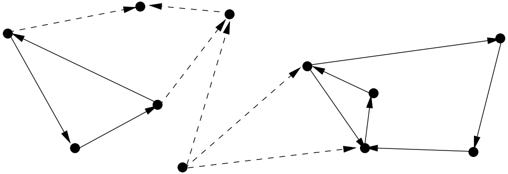
$K_{3,3}$



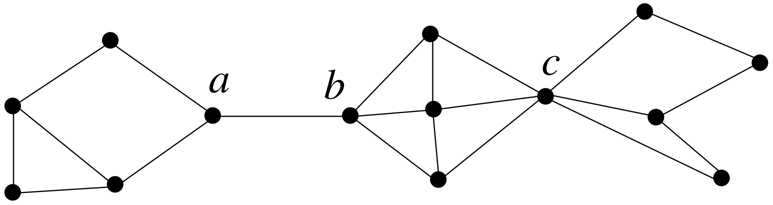
K_5

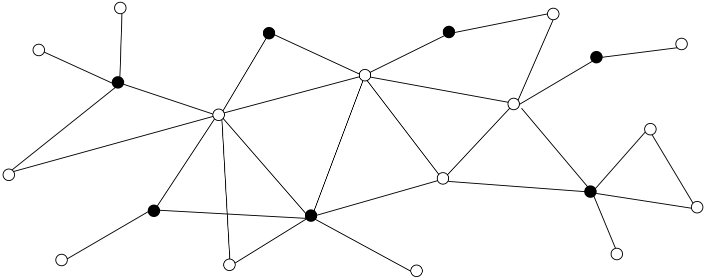


Undirected graph with a clique



Directed graph with two strongly connected components


a
b
c



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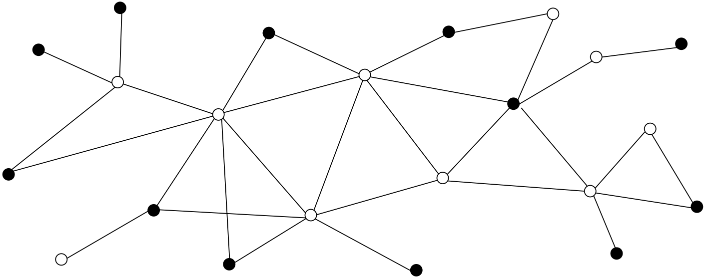

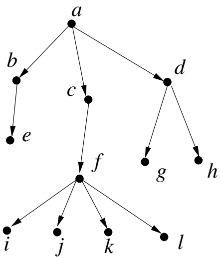
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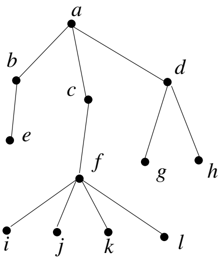
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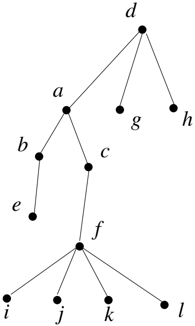
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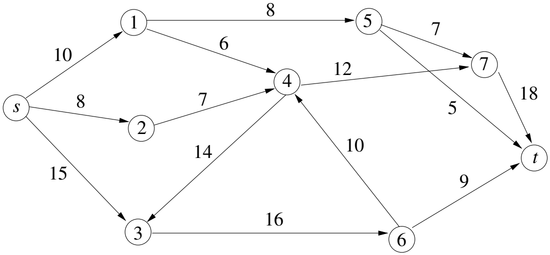
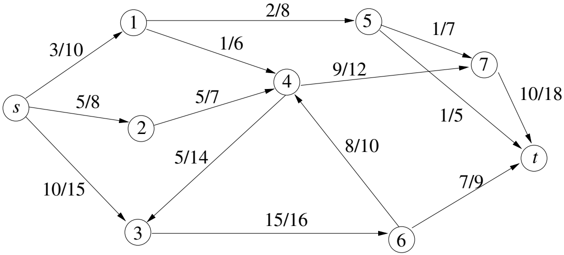
You must transport a cabbage, a goat, and a wolf across a river using a boat. The boat has only enough room for you and one of the other objects. You cannot leave the goat and the cabbage together unsupervised, or the goat will eat the cabbage. Similarly, the wolf will eat the goat if you are not there to prevent it. How can you safely transport all of them to the other side?

fcgw/	*fcg*/*w*	*fw*/*cg*	*c*/*fgw*
cgw/*f*	*gw*/*fc*	*fg*/*cw*	*g*/*fcw*
fgw/*c*	*cw*/*fg*	*fc*/*gw*	*w*/*fcg*
fcw/*g*	*cg*/*fw*	*g*/*cgw*	/*fcgw*



$$\forall \; v \in V - \{s, t\}, \; \sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$$



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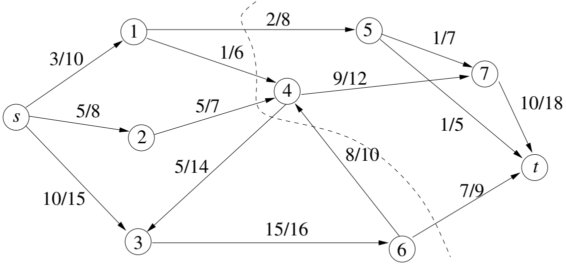

Cut with net flow 2 + 1 + 5 − 5 + 8 + 7 = 18

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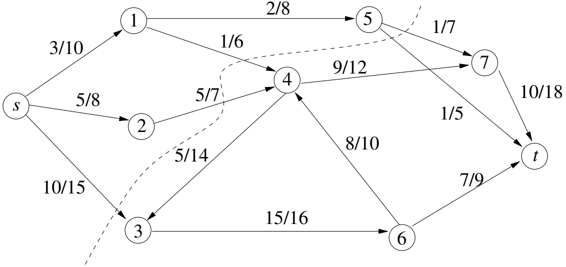
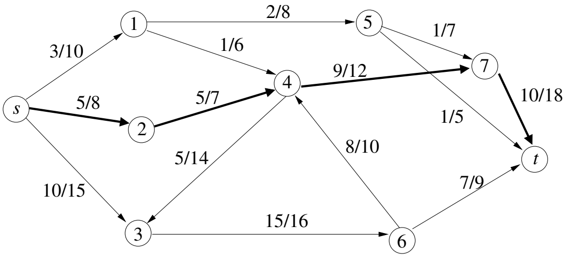


Cut with net flow 1 + 1 + 1 + 5 + 10 = 18



Edge (2, 4) has smallest cut/flow difference

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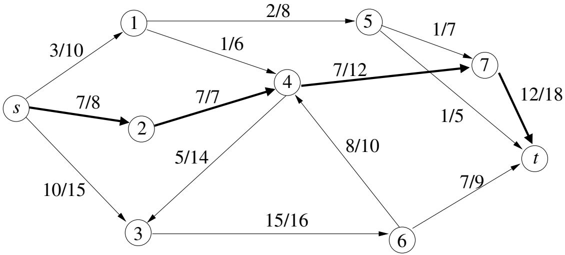

Adding 2 to each edge in path improves the flow.

For Wed, Apr 22:

Review Section 8.5 from DMFP; do Exercises 8.5.(1,2, 3, 4, 10, 11, 24, 33)