Graph unit

- A little more Bayesian inference; begin graph theory (last week Friday)
- Graph proofs (**today**)
- Graph isomorphisms (Wednesday)
- Varieties of graphs (Friday)
- Graphs as models of information (next week Monday)

Today:

- Review graph terms
- Proof graph propositions
- Learn more graph terms
- Prove more graph propositions

A *graph* $G = (V, E)$ is a pair of finite sets, a set V of *vertices* (singular *vertex*) and a set E of pairs of vertices called *edges*. We will typically write $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ where each $e_k = (v_i, v_j)$ for some v_i, v_j; in that case, v_i and v_j are called *end points* of the edge e_k.

An edge (v_i, v_j) is *incident* on its end points v_i and v_j; we also say that it *connects* them. If vertices v_i and v_j are connected by an edge, they are *adjacent* to one another.

If a vertex is adjacent to itself, that connecting edge is called a *self-loop*. If two edges connect the same two vertices, those edges are *parallel* to each other.

A graph is *undirected* if the edges indicate a symmetric relationship between the two endpoints: if there is an edge $e = (v_1, v_2)$, then v_1 is adjacent to v_2 and v_2 is adjacent to v_1; we could have described this edge as (v_2, v_1). In a *directed graph*, sometimes abbreviated to *digraph*, the edges are not symmetric; rather, they have a direction and are displayed with arrowheads.

The *degree* $\deg(v)$ of a vertex v is the number of edges incident on the vertex, with self-loops counted twice. In a directed graph, we need to distinguish between the *in-degree* and *out-degree* of a vertex.

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$ (and, by definition of graph, for any edge $(v_i, v_j) \in F$, $v_i, v_j \in W$).

The *complement* of a simple graph $G = (V, E)$ is a graph $\overline{G} = (V, \overline{E})$ where for $v_i, v_j \in V$, $(v_i, v_j) \in \overline{E}$ if $(v_i, v_j) \notin E$; in other words, the complement has all the same vertices and all (and only) those possible edges that are not in G. A simple graph $G = (V, E)$ is *complete* if for all distinct $v_i, v_j \in V$, the edge $(v_i, v_j) \in E$.

Theorem (Handshake Theorem.)

If $G = (V, E)$ *is an undirected graph with* $V = \{v_1, v_2, \ldots, v_n\}$*, then*

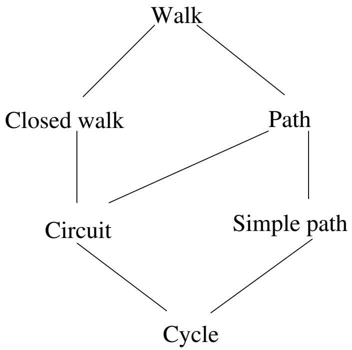
$$\sum_{i=1}^{n} deg(v_i) = 2 \cdot |E|.$$

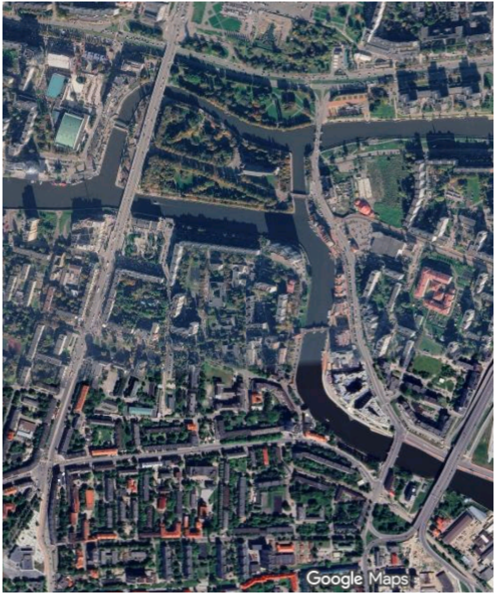
Theorem

If $G = (V, E)$ *is a graph, the total number of vertices with an odd degree are even.*

Adjectives	
Trivial	Having only one vertex and no edges.
Simple	Having no repeated *vertices* (except, possibly, the initial and terminal).
Closed	Having the same vertex as initial and terminal.

Nouns	
Walk	An alternating sequence of vertices and edges, each edge coming between its end points.
Path	A walk with no repeated *edge* (repeated vertices are ok).
Circuit	A closed path (no repeated edges, initial and terminal the same).
Cycle	A simple circuit (no repeated edges or vertices, except the initial and terminal, which are the same).

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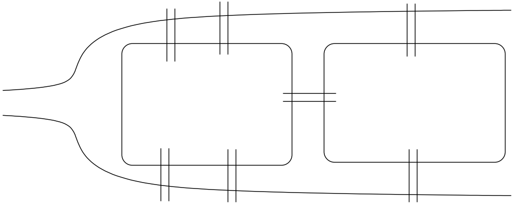
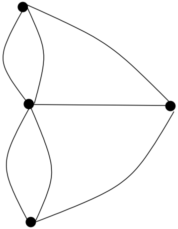


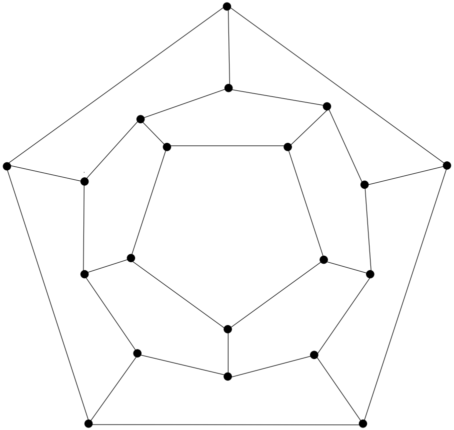


An *Euler circuit* of G is a circuit that contains every vertex and every edge. Since it is a circuit, this also means that an Euler circuit contains every edge exactly once. Vertices, however, may be repeated.

Theorem

If a graph $G = (V, E)$ has an Euler circuit, then every vertex of G has an even degree.

A *Hamiltonian cycle* is a cycle that includes every vertex in V. Since it is a cycle, this means that no vertex or edge is repeated; however, not all the edges need to be included.



For next time:

Read Sections 8.(2 & 3) from DMFP; do Exercises 8.2.(1 & 2)