Discrete probability unit

- Sample spaces, events, axioms or probability (last week Wednesday and Friday)
- The sum and product rules; event-composition method (Monday)
- Conditional, joint, and marginal probabilities (**Today**)
- Independence? (Friday)
- Random variables (next week Monday)
- Review (next week Wednesday)
- Test (next week Friday)

Today:

- Various conditional-probability examples
- Joint and marginal probability decomposition
- Various Conditional-probability results and definitions

Formulas and examples mostly from Ross, *A First Course in Probability*, 1997

A monkey is to demonstrate that she recognizes colors by tossing one red, one black, and one white ball into boxes of the same respective colors, one ball to a box. If the monkey has not learned the colors and merely tosses one ball into each box at random, then find the probabilities of the following results:

a. There are no color matches
b. There is exactly one color match

Wackerly et al, pg 67

Consider a diagnostic test for a rare disease. Let D be the event that a person has the disease, $\overline{D}$ the event that the person does not; let the events that the test returns positive and negative be $+$ and $-$. Suppose that after clinical trials were performed, the rates (probabilities) of events $D \cap +$ etc are

	D	$\overline{D}$
$+$	.009	.099
$-$	.001	.891

How accurate is the test—that is, what is the probability that the test gives the correct result?

If a test returns positive, what is the probability that the patient has the disease?

Let Ω be a sample space and let A and B be events over that sample space. Then $P(A \cap B)$ is the **joint probability** of events A and B.

Suppose $A_0, A_1, \ldots A_{n-1}$ and $B_0, B_1, \ldots B_{m-1}$ are each partitions of sample space Ω. Then the joint probabilities $P(A_i \cap B_j)$ have the property that

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} P(A_i \cap B_j) = 1$$

In this context, the probability of an individual events, for example $P(A_i)$, is called a **marginal probability**, and can be computed by summing over the joint probabilities with the other partition:

$$P(A_i) = \sum_{j=0}^{m-1} P(A_i \cap B_j)$$

	Fair	Good	Very Good	Premium	Ideal		
D	0.0030219	0.0122729	0.0280497	0.0297182	0.0525399	0.1256026	
E	0.0041528	0.017297	0.0444939	0.0433259	0.0723582	0.1816278	
F	0.0057842	0.0168521	0.0401187	0.0432147	0.0709307	0.1769004	
G	0.0058213	0.0161476	0.0426214	0.0542084	0.0905451	0.2093438	
H	0.0056174	0.0130145	0.0338154	0.0437523	0.0577494	0.153949	
I	0.0032443	0.0096774	0.0223211	0.0264739	0.0388024	0.1005191	
J	0.0022062	0.0056915	0.0125695	0.0149796	0.016611	0.0520578	
	0.0298481	0.090953	0.2239897	0.255673	0.3995367		1.0000005
						1.0000005	

Adapted from `https://tinyheero.github.io/2016/03/20/basic-prob.html`

Theorem 1 (Generalization of the product rule)

For all events $A_0, A_1, \ldots A_{n-1}$ over a sample space,

$$P(A_0 \cap A_1 \cap \cdots \cap A_{n-1}) = P(A_0)P(A_2 \mid A_0)P(A_2 \mid A_0 \cap A_1) \cdots P(A_{n-1} \mid A_0 \cap A_1 \cap \cdots \cap A_{n-2})$$

Corollary 2

For all independent events $A_0, A_1, \ldots A_{n-1}$ over a sample space,

$$P(A_0 \cap A_1 \cap \cdots \cap A_{n-1}) = P(A_0)P(A_2) \cdots P(A_{n-1})$$

Suppose that a standard deck of 52 cards is divided into 4 piles of 13 cards each. What is the probability that each pile has exactly one ace?

Theorem 3 (Bayes's formula)

For any events A and B over a sample space,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Or, more generally, for any partition of the sample space $A_0, A_1, \ldots A_{n-1}$,

$$P(A_j \mid B) = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=0}^{n-1} P(B \mid A_i)P(A_i)}$$

A game show features an event where a contestant is shown three doors, with the grand prize hidden behind one of them. The contestant picks one door. To raise the drama, the host then opens one of the other doors, showing it to be empty, and gives the contestant the option of switching to the remaining closed door. Strategically, should the contestant switch?

For next time:

Take the quiz (on Canvas)