

Probability and stats mop-up unit

- ▶ Jointly distributed random variables (spring-break eve)
- ▶ Understanding covariance and correlation (Monday)
- ▶ Convergence of random variables (Wednesday)
- ▶ The weak law of large numbers (**Today**)
- ▶ The central limit theorem (next week Monday)
- ▶ Review for test (next week Wednesday)
- ▶ Test 2 (next week Friday)

Today:

- ▶ Limits
- ▶ Sequences of random variables
- ▶ Kinds of convergence
- ▶ The Weak Law of Large Numbers

Theorem (Markov's inequality)

If X is a random variable with codomain $\mathbb{R}^{\text{nonneg}}$, then for any $a \in \mathbb{R}^+$,

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Theorem (Chebyshev's inequality)

If X is a random variable with mean μ and variance σ^2 , then for any $k \in \mathbb{R}^+$,

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

The formal definition of a limit:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and let $a, L \in \mathbb{R}$. Then the limit of f as x approaches a is L , written as

$$\lim_{x \rightarrow a} f(x) = L$$

if for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The most important aspect of probability theory concerns the behavior of sequences of random variables. . . . The basic question is this: what can we say about the limiting behavior of a sequence of random variables $X_0, X_1, X_2 \dots$? Since statistics and data mining are all about gathering data, we will naturally be interested in what happens as we gather more and more data.

Wasserman, All of Statistics, 2004

The most important theoretical results in probability theory are limit theorems. Of these the most important are those that are classified either under the heading “laws of large numbers” or under the heading “central limit theorems.” Usually, theorems are considered to be laws of large numbers if they are concerned with stating conditions under which the average of a sequence of random variables converges (in some sense) to the expected average. On the other hand, central limit theorems are concerned with determining conditions under which the sum of a large number of random variables has a probability distribution that is approximately normal.

Ross, A First Course in Probability, 1997

Let X_0, X_1, \dots, X_n be random variables over the same sample space. As an ordered collection, we refer to them as a **sequence of random variables** of size n .

The **sample mean** of this sequence is

$$\bar{X}_n = \frac{1}{n} \sum_{i=0}^{n-1} X_i$$

If each random variable X_i has the same distribution and they are independent, we refer to them as **independent and identically-distributed** (IID) random variables.

Let X_0, X_1, \dots be a sequence of random variables and let X be another random variable. Let F_n be the cumulative distribution function of X_n and F be the cumulative distribution function of X .

X_n **converges to X in probability**, written as $X_n \xrightarrow{\mathcal{P}} X$, if for any $\epsilon \in \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$$

X_n **converges to X in distribution**, written as $X_n \xrightarrow{\mathcal{D}} X$, if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x)$$

X_n **converges to X in quadratic mean**, written as $X_n \xrightarrow{\mathcal{QM}} X$, if

$$\lim_{n \rightarrow \infty} E[(X_n - X)^2] = 0$$

X_n **converges almost surely** to X , written as $X_n \xrightarrow{\mathcal{AS}} X$, if

$$P(\lim_{n \rightarrow \infty} X_n = X) = 1$$

Theorem

If X_0, X_1, \dots be a sequence of IID random variables, $\mu = E[X_i]$, and $\sigma^2 = \text{Var}(X_i)$, then

a. $E[\bar{X}_n] = \mu$

b. $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$

Theorem (The Weak Law of Large Numbers)

If X_0, X_1, \dots, X_{n-1} are a sequence of independent and identically-distributed random variables, each having mean $E[X_i] = \mu$, then

$$\bar{X}_n \xrightarrow{\mathcal{P}} \mu$$

that is, the sample mean converges in probability to μ .

For next time:

Take quiz on Canvas