Sets and Elements

Slides to accompany Sections 1.(1-3) of *Discrete Mathematics and Functional Programming*

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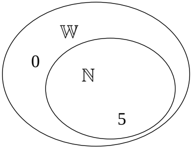
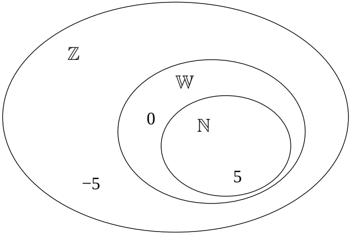
Categories of numbers



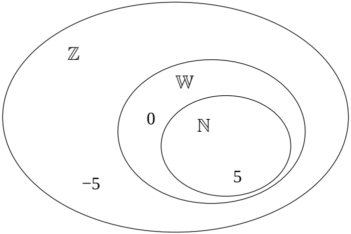
Natural numbers and *whole numbers*.

Categories of numbers



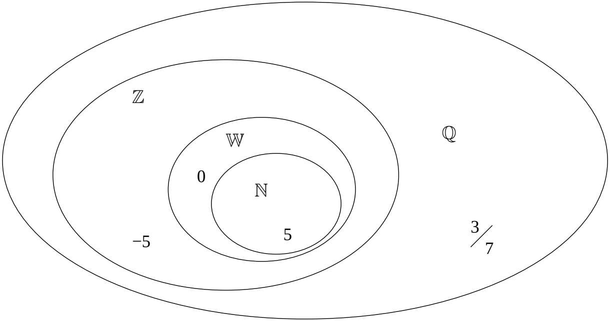
Adding *integers*.

Categories of numbers



Adding *integers*.

Categories of numbers

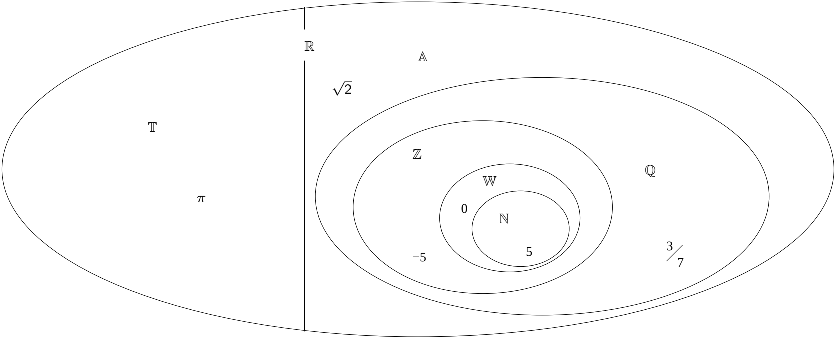


Adding *rationals*.

Categories of numbers

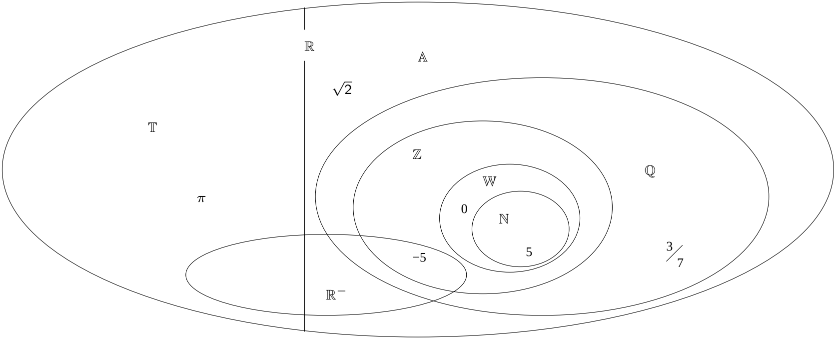
Measurable real-world lengths that are not rationals.

Categories of numbers



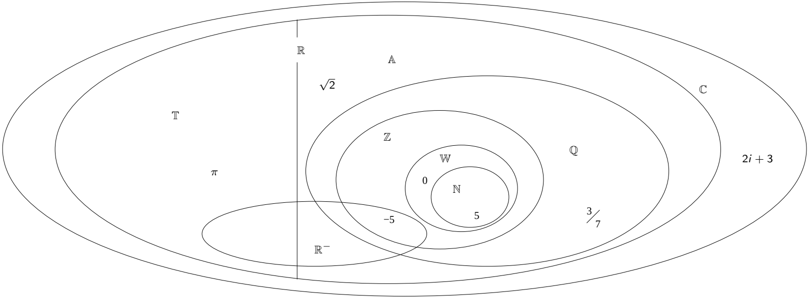
Differentiating *algebraic numbers* from *transcendental numbers*.

Categories of numbers



Differentiating *negative numbers*.

Categories of numbers



Adding *complex numbers*.

Categories of numbers

5 is a natural number (*or* the collection of natural numbers contains 5).	$5 \in \mathbb{N}$
All integers are rational numbers.	$\mathbb{Z} \subseteq \mathbb{Q}$
Merging the algebraic numbers and the transcendental numbers makes the real numbers.	$\mathbb{R} = \mathbb{A} \cup \mathbb{T}$
Negative integers are both negative and integers.	$\mathbb{Z}^- = \mathbb{R}^- \cap \mathbb{Z}$
Transcendental numbers are those real numbers that are not algebraic numbers.	$\mathbb{T} = \mathbb{R} - \mathbb{A}$

Categories of numbers

Nothing is both transcendental and algebraic, *or* the collection of things both transcendental and algebraic is empty.

$$\mathbb{T} \cap \mathbb{A} = \emptyset$$

Adding 0 to the collection of natural numbers makes the collection of whole numbers.

$$\mathbb{W} = \{0\} \cup \mathbb{N}$$

Since all rational numbers are algebraic numbers and all algebraic numbers are real numbers, it follows that all rational numbers are real numbers.

$$\begin{aligned} & \mathbb{Q} \subseteq \mathbb{A} \\ & \mathbb{A} \subseteq \mathbb{R} \\ \therefore \quad & \mathbb{Q} \subseteq \mathbb{R} \end{aligned}$$

Primitive terms

Things like 5, $\frac{3}{7}$, $\sqrt{2}$, π, $2i + 3$ are *elements*.

Things like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are *sets*.

Primitive terms are ideas too basic to define; we use them in defining other terms. *Set* and *element* are primitive terms of set theory just as *point*, *line*, and *plane* are primitive terms of geometry.

Primitive terms and axioms

Instead of defining primitive terms, we define their relationships among them. Here are two axioms of set theory (a complete axiomatic formulation requires many more):

Axiom (Existence.)

There is a set with no elements.

Axiom (Extensionality.)

If every element of a set X is an element of a set Y and every element of Y is an element of X, then $X = Y$.

Set notation

The empty set: $\emptyset$ or $\{\}$

Explicit listing of elements (sets are unordered, so order doesn't matter):

$$\begin{aligned}\{\text{Red}, \text{Green}, \text{Blue}\} &= \{\text{Green}, \text{Blue}, \text{Red}\} = \{\text{Blue}, \text{Red}, \text{Green}\} \\ &= \{\text{Blue}, \text{Green}, \text{Red}\}\end{aligned}$$

Membership of an element:

$$\text{Red} \in \{\text{Green}, \text{Red}\}$$

Set notation

Defining a set as a restriction on a larger set:

$$\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$$

This means the set of natural numbers is the set of elements x drawn from integers with the restriction that $x > 0$.

Other ways to define sets exist, such as “interval” notation you saw in high school or earlier:

$$(1, 5] = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$$