Set operations and facts about sets

Slides to accompany Sections 1.(4 & 5) of *Discrete Mathematics and Functional Programming*

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Operations from arithmetic

These operations on numbers produce new numbers. Grammatically, they are equivalent to nouns.

$$5 + 3 \qquad 12 - 7 \qquad (18 \cdot 13) \div 21$$

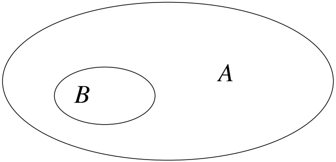
These operations produce a true or false value. Grammatically, they are equivalent to declarative sentences.

$$5 + 3 = 8 \qquad 17 > 18 \div 6 \qquad (15 + 4) \cdot 21 \leq 3 - 2$$

Operations on sets

We have two main sentence-making operations for sets:
$A = B$, meaning A and B have exactly the same elements.
$B \subseteq A$ meaning every element in B is an element in A; B is a *subset* of A.

$B \subseteq A$



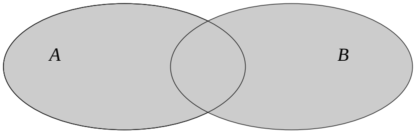
Also we have *proper subset* $B \subset A$, meaning $B \subseteq A$ but $B \neq A$, or at least one element of A isn't in B. Similarly we have *superset* $B \supseteq A$ and *proper superset* $B \supset A$. These aren't used very often, but $\subseteq, \subset, \supseteq, \supset$ are analogous to $\leq, <, \geq, >$.

Set-making operations: Union

We have three operations on sets that result in new sets. The *union* of two sets is the set of elements that are in **either** set.

$$A \cup B = \{\ x \mid x \in A \text{ or } x \in B\}$$

$$\begin{aligned} \{1,2,3\} \cup \{2,3,4\} &= \{1,2,3,4\} \\ \{1,2\} \cup \{3,4\} &= \{1,2,3,4\} \\ \{1,2\} \cup \{1,2,3\} &= \{1,2,3\} \end{aligned}$$

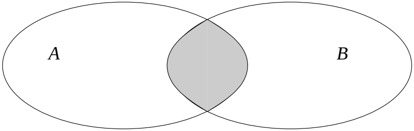


Set-making operations: Intersection

The *intersection* of two sets is the set of elements that are in **both** sets.

$$A \cap B = \{ \ x \mid x \in A \text{ and } x \in B\}$$

$$\begin{aligned} \{1,2,3\} \cap \{2,3,4\} &= \{2,3\} \\ \{1,2\} \cap \{3,4\} &= \emptyset \\ \{1,2\} \cap \{1,2,3\} &= \{1,2\} \end{aligned}$$

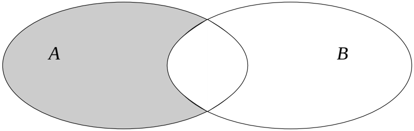


Set-making operations: Difference

The *difference* of two sets is the set of elements that are in the **first** set but **not** in the **second**.

$$A - B = \{\ x \mid x \in A \text{ and } x \notin B\}$$

$$\begin{aligned} \{1,2,3\} - \{2,3,4\} &= \{1\} \\ \{1,2\} - \{3,4\} &= \{1,2\} \\ \{1,2\} - \{1,2,3\} &= \emptyset \end{aligned}$$



Set-making operations: All together

Union

The set of elements that are in **either** set.

$$A \cup B = \{\ x \mid x \in A \text{ or } x \in B\}$$

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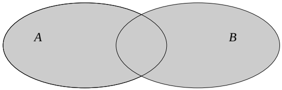

Intersection

The set of elements that are in **both** sets.

$$A \cap B = \{\ x \mid x \in A \text{ and } x \in B\}$$

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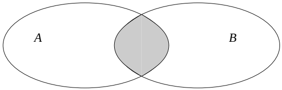
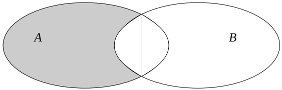


Difference

The set of elements that are in the **first** set but **not** in the **second**.

$$A - B = \{\ x \mid x \in A \text{ and } x \notin B\}$$

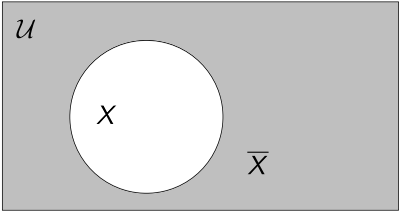
$$\begin{aligned} \{1,2,3\} - \{2,3,4\} &= \{1\} \\ \{1,2\} - \{3,4\} &= \{1,2\} \\ \{1,2\} - \{1,2,3\} &= \emptyset \end{aligned}$$



Set complement

The *universal set*, $\mathcal{U}$, is the set of all elements under discussion. This allows us to define the *complement* of a set, the set of everything not in given set:

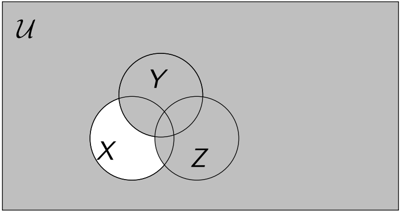
$$\overline{X} = \{x \in \mathcal{U} \mid x \notin X\}$$



Complement is the analogue of negation (that is, the negative sign) in arithmetic. They are both *unary* operators, which means they take only one parameter.

Combining set operations

Set operations can be arbitrarily combined.



$$\overline{X - (Y \cup Z)}.$$

Observations about set operations

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{5, 6, 7\}$.

$$\begin{aligned} A \cup (B \cap C) &= \{1, 2, 3\} \cup (\{3, 4, 5\} \cap \{5, 6, 7\}) \\ &= \{1, 2, 3\} \cup \{5\} \\ &= \{1, 2, 3, 5\} \end{aligned}$$

and

$$\begin{aligned} (A \cup B) \cap (A \cup C) &= (\{1, 2, 3\} \cup \{3, 4, 5\}) \cap (\{1, 2, 3\} \cup \{5, 6, 7\}) \\ &= \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 5, 6, 7\} \\ &= \{1, 2, 3, 5\} \end{aligned}$$

In other words, for these sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Hypotheses about set operations

We suspect that for *any* three sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

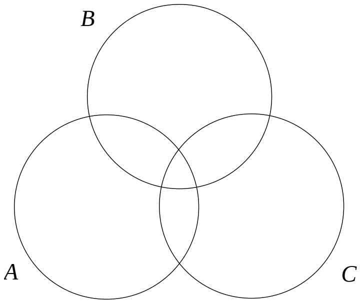
This would be a *distributive law*, analogous to the distributive law of arithmetic you learned in grade school:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is also true. . . see Exercise 1.5.4.

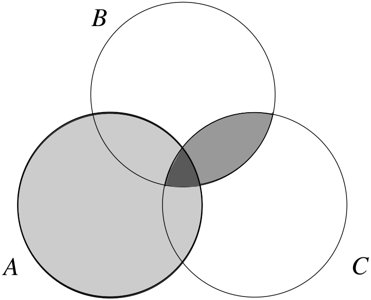
Facts about set operations

A large part of this course is about proving facts about sets formally. Before we get to writing proofs, we can verify facts like this informally using Venn diagrams.
Start with a blank template.



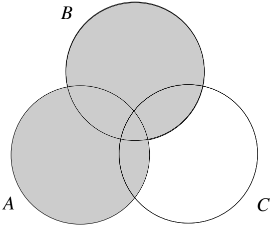
Verifying facts about sets

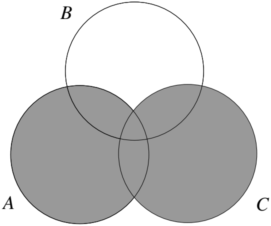
Shade A with and $B \cap C$ with . The overlap $A \cap (B \cap C)$ has the darkest tint ,



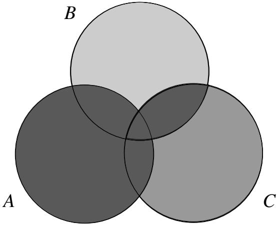
Verifying facts about sets

Separately, superimpose $A \cup B$ shaded and $A \cup C$ shaded .





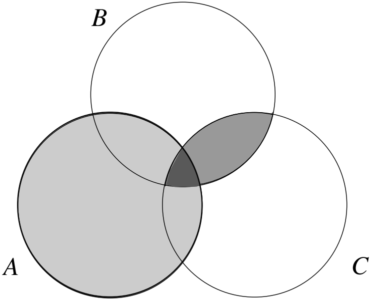
To get



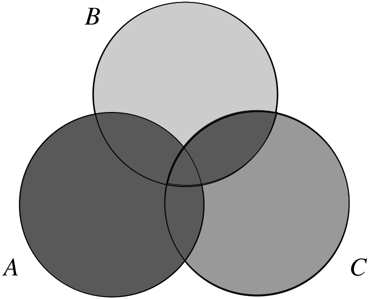
The overlap $(A \cap B) \cup (A \cap C)$ is shaded

Verifying facts about sets

Put together, we see that *anything shaded* on the left matches the *darkly* (or *double*) *shaded* on the right.



$A \cap (B \cup C)$
(Any shade)



$(A \cap B) \cup (A \cap C)$
(Double shade)

Verifying facts about sets

Another example:

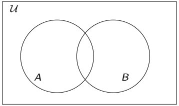
$$\overline{A} \cup B = \overline{A - B}$$

Intuition: Alvin, Beverley, Camus, Daisy, Eddie, and Gladys are cattle. Let A be the set of cows. $A = \{\text{Beverley}, \text{Daisy}, \text{Gladys}\}$. Let $B = \{\text{Alvin}, \text{Beverley}, \text{Camus}, \text{Gladys}\}$ be the spotted ones.

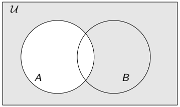
$$\begin{aligned}
\text{Bulls or spotted:} \quad \overline{A} \cup B &= \overline{\{\text{Beverley}, \text{Daisy}, \text{Gladys}\}} \cup \{\text{Alvin}, \text{Beverley}, \text{Camus}, \text{Gladys}\} \\
&= \{\text{Alvin}, \text{Camus}, \text{Eddie}\} \cup \{\text{Alvin}, \text{Beverley}, \text{Camus}, \text{Gladys}\} \\
&= \{\text{Alvin}, \text{Beverley}, \text{Camus}, \text{Eddie}, \text{Gladys}\} \\
&= \overline{\{\text{Daisy}\}} \\
&= \overline{\{\text{Beverley}, \text{Daisy}, \text{Gladys}\} - \{\text{Alvin}, \text{Beverley}, \text{Camus}, \text{Gladys}\}} \\
&= \overline{A - B} \qquad : \text{All but spotted cows}
\end{aligned}$$

Verifying facts about sets

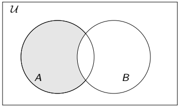
Visually:



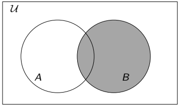
Original



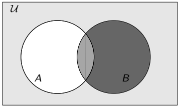
$\overline{A}$



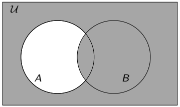
$A - B$



B



$\overline{A} \cup B$ any shade



$\overline{A - B}$