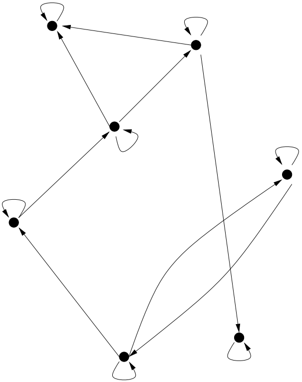
Properties of relations

Slides to accompany Section 5.4 of *Discrete Mathematics and Functional Programming*

Thomas VanDrunen

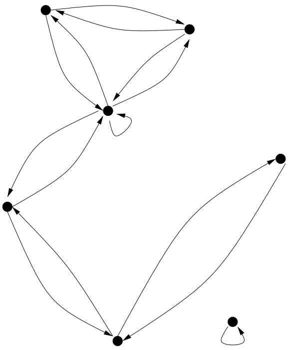
Reflexivity



A relation R on a set X is *reflexive* if every element is related to itself:

$$\forall\, x \in X, (x,x) \in R$$

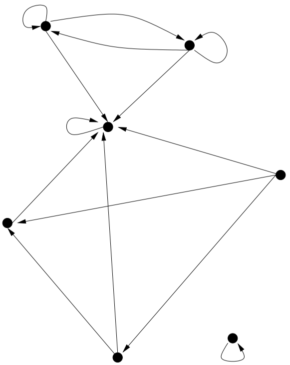
Symmetry



A relation R on a set X is *symmetric* if for every pair in the relation, the inverse of the pair also exists:

$$\forall\ x, y \in X, \quad \text{if} \ \ (x, y) \in R \quad \text{then} \ \ (y, x) \in R$$

Transitivity

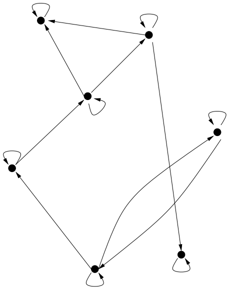


A relation R on a set X is *transitive* if any time one element is related to a second and that second is related to a third, then the first is also related to the third:

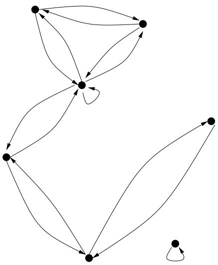
$$\forall\, x, y, z \in X, \quad \text{if } (x, y) \in R \text{ and } (y, z) \in R, \text{ then } (x, z) \in R$$

Summary

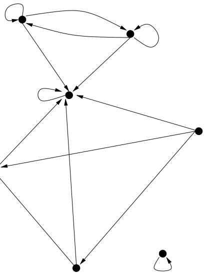
$\forall\ x \in X, (x, x) \in R$



$\forall\ x, y \in X,\quad$ if $(x, y) \in R$ then $(y, x) \in R$



$\forall\ x, y, z \in X,\quad$ if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$



Proof patterns

$\forall\, x \in X, (x,x) \in R$

Suppose $x \in X$.
...
Hence $(x,x) \in R$.
Therefore R is reflexive. □

$\forall\, x, y \in X,$ if $(x,y) \in R$ then $(y,x) \in R$

Suppose $x, y \in X$. Further suppose $(x,y) \in R$.
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Hence $(y,x) \in R$.
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$\forall\, x, y, z \in X,$ if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

Suppose $x, y, z \in X$. Further suppose $(x,y) \in R$ and $(y,z) \in R$.
...
Hence $(x,z) \in R$.
Therefore R is transitive. □

Proof patterns—short versions

$\forall\, x \in X, (x,x) \in R$

Suppose $x \in X$.

...

Hence $(x,x) \in R$.

Therefore R is reflexive. $\square$

$\forall\, x, y \in X,$ if $(x,y) \in R$ then $(y,x) \in R$

Suppose $(x,y) \in R$.

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$\forall\, x, y, z \in X,$ if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

Suppose $(x,y) \in R$ and $(y,z) \in R$.

...

Hence $(x,z) \in R$.

Therefore R is transitive. $\square$

Reflexivity example

Proposition 1

The relation $\mid$ *on* $\mathbb{N}$ *is reflexive.*

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Proof. *Suppose $a \in \mathbb{N}$.*

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Reflexivity example

Proposition 1

The relation $|$ on $\mathbb{N}$ is reflexive.

Proof. *Suppose $a \in \mathbb{N}$.*
By arithmetic $a \cdot 1 = a$, and so by the definition of divides, $a|a$.
Hence, by the definition of reflexive, $|$ is reflexive. $\square$

Symmetry example

Proposition 2

The relation "is opposite of" on $\mathbb{Z}$ is symmetric.

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Proof. *Suppose* $x, y \in \mathbb{Z}$.

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Proof. *Suppose $x, y \in \mathbb{Z}$. Further suppose $x + y = 0$.*

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The relation "is opposite of" on $\mathbb{Z}$ is symmetric.

Proof. *Suppose* $x, y \in \mathbb{Z}$*. Further suppose* $x + y = 0$*.*
$y + x = x + y$ *by commutativity of addition.*

Symmetry example

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The relation "is opposite of" on $\mathbb{Z}$ is symmetric.

Proof. *Suppose* $x, y \in \mathbb{Z}$. *Further suppose* $x + y = 0$. $y + x = x + y$ *by commutativity of addition.* $y + x = 0$ *by substitution.*

Symmetry example

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The relation "is opposite of" on $\mathbb{Z}$ is symmetric.

Proof. *Suppose* $x, y \in \mathbb{Z}$. *Further suppose* $x + y = 0$.
$y + x = x + y$ *by commutativity of addition.* $y + x = 0$ *by substitution.*
Therefore "is opposite of" is symmetric. □

Transitivity example

Proposition 3

The relation $\mid$ on $\mathbb{Z}$ is transitive.

Transitivity example

Proposition 3

The relation $|$ on $\mathbb{Z}$ is transitive.

Proof. *Suppose* $a, b, c \in \mathbb{Z}$, *and suppose* $a|b$ *and* $b|c$.

Transitivity example

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The relation $|$ on $\mathbb{Z}$ is transitive.

Proof. *Suppose $a|b$ and $b|c$.*

Transitivity example

Proposition 3

The relation $|$ on $\mathbb{Z}$ is transitive.

Proof. *Suppose* $a|b$ *and* $b|c$.
By the definition of divides, there exist $d, e \in \mathbb{Z}$ *such that* $a \cdot d = b$ *and* $b \cdot e = c$.

Transitivity example

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The relation $|$ on $\mathbb{Z}$ is transitive.

Proof. *Suppose $a|b$ and $b|c$.*
By the definition of divides, there exist $d, e \in \mathbb{Z}$ such that $a \cdot d = b$ and $b \cdot e = c$. By substitution and associativity, $a(d \cdot e) = c$.

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By the definition of divides, there exist $d, e \in \mathbb{Z}$ such that $a \cdot d = b$ and $b \cdot e = c$. By substitution and associativity, $a(d \cdot e) = c$.
By the definition of divides, $a|c$.

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By the definition of divides, there exist $d, e \in \mathbb{Z}$ such that $a \cdot d = b$ and $b \cdot e = c$. By substitution and associativity, $a(d \cdot e) = c$.
By the definition of divides, $a|c$. Hence $|$ is transitive. $\square$

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Proposition 4

If R is reflexive, then $i_A \subseteq R$.

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Proof. *Suppose R is reflexive.*

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Proof. *Suppose R is reflexive. Further suppose that $(a, b) \in i_A$.*

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By definition of identity relation, $a = b$.

Reflexivity example

Proposition 4

If R is reflexive, then $i_A \subseteq R$.

Proof. *Suppose R is reflexive. Further suppose that $(a, b) \in i_A$.*
By definition of identity relation, $a = b$. By definition of reflexivity, since R is reflexive, $(a, b) \in R$.

Reflexivity example

Proposition 4

If R is reflexive, then $i_A \subseteq R$.

Proof. *Suppose R is reflexive. Further suppose that $(a, b) \in i_A$.*
By definition of identity relation, $a = b$. By definition of reflexivity, since R is reflexive, $(a, b) \in R$.
Therefore, by definition of subset, $i_A \subseteq R$. $\square$

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Proposition 5

If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

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Proof. *Suppose R is a relation on a set A. Next, suppose $a, b \in A$.*

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Proof. *Suppose R is a relation on a set A. Next, suppose $a, b \in A$. Finally, suppose $(a, b) \in R \cap R^{-1}$.*

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If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

Proof. *Suppose R is a relation on a set A. Further suppose $(a, b) \in R \cap R^{-1}$.*

Symmetry example

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If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

Proof. *Suppose R is a relation on a set A. Further suppose $(a, b) \in R \cap R^{-1}$.*
By definition of intersection, $(a, b) \in R$ and $(a, b) \in R^{-1}$.

Symmetry example

Proposition 5

If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

Proof. *Suppose R is a relation on a set A. Further suppose $(a, b) \in R \cap R^{-1}$.*
By definition of intersection, $(a, b) \in R$ and $(a, b) \in R^{-1}$.
Since $(a, b) \in R$, the definition of inverse tells us that $(b, a) \in R^{-1}$. Similarly, since $(a, b) \in R^{-1}$, by definition of inverse it is also the case that $(b, a) \in R$.

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If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

Proof. *Suppose R is a relation on a set A. Further suppose $(a, b) \in R \cap R^{-1}$.*
By definition of intersection, $(a, b) \in R$ and $(a, b) \in R^{-1}$.
Since $(a, b) \in R$, the definition of inverse tells us that $(b, a) \in R^{-1}$. Similarly, since $(a, b) \in R^{-1}$, by definition of inverse it is also the case that $(b, a) \in R$.
By definition of intersection, $(b, a) \in R \cap R^{-1}$.

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If R is a relation on a set A, then $R \cap R^{-1}$ is symmetric.

Proof. *Suppose R is a relation on a set A. Further suppose $(a, b) \in R \cap R^{-1}$.*
By definition of intersection, $(a, b) \in R$ and $(a, b) \in R^{-1}$.
Since $(a, b) \in R$, the definition of inverse tells us that $(b, a) \in R^{-1}$. Similarly, since $(a, b) \in R^{-1}$, by definition of inverse it is also the case that $(b, a) \in R$.
By definition of intersection, $(b, a) \in R \cap R^{-1}$. Therefore $R \cap R^{-1}$ is symmetric by definition. $\square$

Transitivity example

Proposition 6

If R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$, then R is transitive.

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Proof. *Suppose R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$.*

Transitivity example

Proposition 6

If R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$, then R is transitive.

Proof. *Suppose R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$.*
Further suppose that $(b, c), (c, d) \in R$.

Transitivity example

Proposition 6

If R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$, then R is transitive.

Proof. *Suppose R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$.*
Further suppose that $(b, c), (c, d) \in R$.
By definition of image, $c \in \mathcal{I}_R(b)$.

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If R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$, then R is transitive.

Proof. *Suppose R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$.*
Further suppose that $(b, c), (c, d) \in R$.
By definition of image, $c \in \mathcal{I}_R(b)$. By definition of image, $d \in \mathcal{I}_R(\mathcal{I}_R(b))$

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Further suppose that $(b, c), (c, d) \in R$.
By definition of image, $c \in \mathcal{I}_R(b)$. By definition of image, $d \in \mathcal{I}_R(\mathcal{I}_R(b))$ By definition of subset, $d \in \mathcal{I}_R(b)$.

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If R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$, then R is transitive.

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Further suppose that $(b, c), (c, d) \in R$.
By definition of image, $c \in \mathcal{I}_R(b)$. By definition of image, $d \in \mathcal{I}_R(\mathcal{I}_R(b))$ By definition of subset, $d \in \mathcal{I}_R(b)$.
By definition of image, $(b, d) \in R$.

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Proof. *Suppose R is a relation on A and for all $a \in A$, $\mathcal{I}_R(\mathcal{I}_R(a)) \subseteq \mathcal{I}_R(a))$.*
Further suppose that $(b, c), (c, d) \in R$.
By definition of image, $c \in \mathcal{I}_R(b)$. By definition of image, $d \in \mathcal{I}_R(\mathcal{I}_R(b))$ By definition of subset, $d \in \mathcal{I}_R(b)$.
By definition of image, $(b, d) \in R$.
Therefore R is transitive by definition. □

Proof patterns

$\forall\, x \in X, (x,x) \in R$

Suppose $x \in X$.
...
Hence $(x,x) \in R$.
Therefore R is reflexive. $\square$

$\forall\, x,y \in X$, if $(x,y) \in R$ then $(y,x) \in R$

Suppose $x,y \in X$. Further suppose $(x,y) \in R$.
...
Hence $(y,x) \in R$.
Therefore R is symmetric. $\square$

$\forall\, x,y,z \in X$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

Suppose $x,y,z \in X$. Further suppose $(x,y) \in R$ and $(y,z) \in R$.
...
Hence $(x,z) \in R$.
Therefore R is transitive. $\square$

Properties of relations

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