Unless otherwise noted, assume $f : X \to Y$.

5.4.6. If $A, B \subseteq Y$, then $F^{-1}(A \cap B) = F^{-1}(A) \cap F^{-1}(B)$.

> **Proof.** Suppose $A, B \subseteq Y$.
>
> First, suppose $x \in F^{-1}(A \cap B)$. By definition of inverse image, $f(x) \in A \cap B$. By definition of intersection, $f(x) \in A$ and $f(x) \in B$. By definition of inverse image, $x \in F^{-1}(A)$ and $x \in F^{-1}(B)$. By definition of intersection, $x \in F^{-1}(A) \cap F^{-1}(B)$. Hence, by definition of subset, $F^{-1}(A \cap B) \subseteq F^{-1}(A) \cap F^{-1}(B)$.
>
> The other direction is symmetric. $\square$

5.6.3. If $A \subseteq X$ and f is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

> **Proof.** Suppose $A \subseteq X$ and f is one-to-one.
>
> Further Suppose $x \in F^{-1}(F(A))$. By definition of inverse image, $f(x) \in F(A)$. By definition of image, there exists $a \in A$ such that $f(a) = f(x)$. [**Note that we do not yet know $x \in A$! The fact that $f(x) \in F(A)$ only gives us that there exists some $a \in A$ such that $f(a) = f(x)$.**]
>
> By definition of one-to-one, $x = a$. By substitution, $x \in A$. By definition of subset, $F^{-1}(F(A)) \subseteq A$. $\square$.

If you find a slightly longer version more clear, here it is:

> **Proof.** Suppose $A \subseteq X$ and f is one-to-one.
>
> Further Suppose $x \in F^{-1}(F(A))$. By definition of inverse image, $f(x) \in F(A)$. Let $y = f(x)$. By definition of image, there exists $a \in A$ such that $f(a) = y$. By substitution, $f(a) = f(x)$.
>
> By definition of one-to-on, $x = a$. By substitution, $x \in A$. By definition of subset, $F^{-1}(F(A)) \subseteq A$. $\square$.

5.6.4. If $A \subseteq Y$ and f is onto, then $A \subseteq F(F^{-1}(A))$.

> **Proof.** Suppose $A \subseteq Y$ and f is onto.
>
> Further suppose $y \in A$. By definition of onto, there exists $x \in X$ such that $f(x) = y$. By definition of inverse image, $x \in F^{-1}(y)$. By definition of image, $y \in F(F^{-1})(y)$. $\square$

5.8.3. If $f : A \to B$, $g : B \to C$, $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Suppose $f : A \to B$, $g : B \to C$, $h : C \to D$.

Suppose $a \in A$. Then

$$\begin{array}{rcll} h \circ (g \circ f)(a) & = & h(g \circ f(a)) & \text{by definition of function composition} \\ & = & h(g(f(a))) & \text{"} \quad \text{"} \\ & = & h \circ g(f(a)) & \text{"} \quad \text{"} \\ & = & (h \circ g) \circ f(a) & \text{"} \quad \text{"} \end{array}$$

Therefore, by definition of function equality, $h \circ (g \circ f) = (h \circ g) \circ f$. □

5.8.4. If $f : A \to B$ and $g : B \to C$ are both onto, then $g \circ f$ is also onto.

Proof. Suppose $f : A \to B$ and $g : B \to C$ are both onto.

Suppose $c \in C$. By definition of onto (since g is onto), there exists a $b \in B$ such that $g(b) = c$. Similarly (since f is onto), there exists an $a \in A$ such that $f(a) = b$.

By substitution, $g(f(a)) = c$. By definition of function composition, $g \circ f(a) = c$.

Therefore, by definition of onto, $g \circ f$ is onto. □

5.8.5. If $f : A \to B$, $g : B \to C$, and $g \circ f$ is one-to-one, then f is one-to-one.

Proof. Suppose $f : A \to B$, $g : B \to C$, and $g \circ f$ is one-to-one.

Further suppose $x, y \in A$ such that $f(x) = f(y)$. By substitution, $g(f(x)) = g(f(y))$. By definition of function composition, $g \circ f(x) = g \circ f(y)$. By definition of one-to-one (since $g \circ f$ is one-to-one), $x = y$.

Therefore, by definition of one-to-one, f is one-to-one. □