Chapter 7 outline:

- ▶ Introduction, function equality, and anonymous functions (last Friday)
- ▶ Image and inverse images (Monday)
- ▶ Function properties, composition, and applications to programming (**Today**)
- ▶ Cardinality (Friday)
- ▶ *Practice quiz* and Countability (next week Monday)
- ▶ Review (Monday, Nov 29)
- ▶ Test 3, on Ch 6 & 7 (Wednesday, Dec 1)

Today:

- ▶ Programming: map and filter
- ▶ Definition of one-to-one and onto, plus proofs
- ▶ Inverse functions
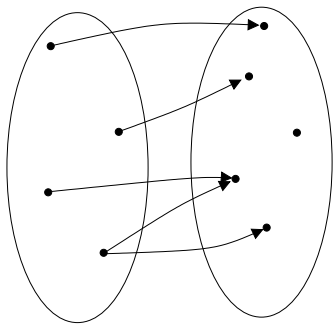- ▶ Definition of function composition, plus proofs

For next time:

Pg 346: 7.6.(2, 3, 6)
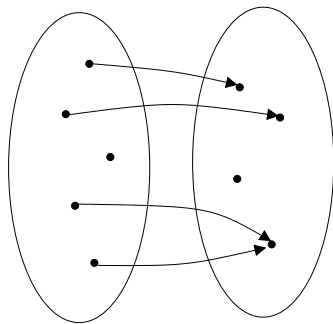
Ex "7.5.(a-c)" on Schoology
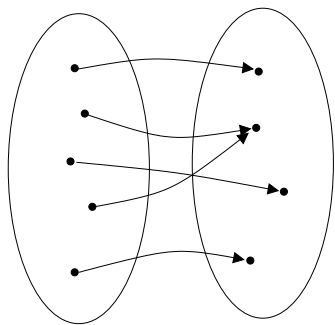
Pg 351: 7.8.(1, 5, 6)

Skim 7.9

Not a function.
(There's a domain element that is related to two things.)

Not a function.
(There's a domain element that is not related to anything.)

Onto (Surjection)

Everything in the codomain
is hit.

$f : X \to Y$ is onto if $\forall y \in Y$,
  $\exists\, x \in X \mid f(x) = y$.

Analytic use:
f is onto.
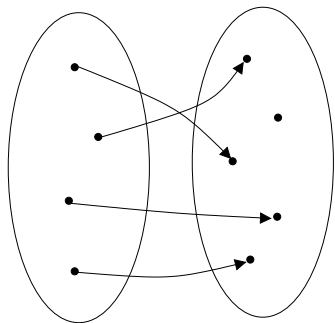$y \in Y$.
Hence $\exists\, x \in X$ such that $f(x) = y$.

Synthetic use:
Suppose $y \in Y$.
$\vdots$
(Somehow find x such that $f(x) = y$.)
Therefore f is onto.

One-to-one (Injection)

Domain elements don't share.

f is one-to-one if $\forall\ x_1, x_2 \in X$,
    if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Analytic use:
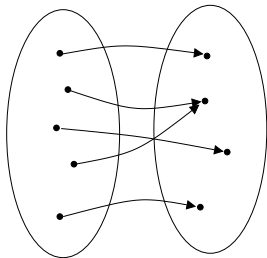f is one-to-one.
$f(x_1) = f(x_2)$.
Hence $x_1 = x_2$.

Synthetic use:
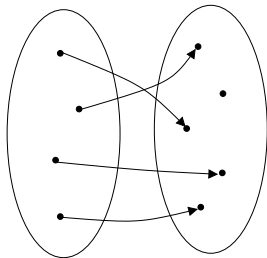Suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$.
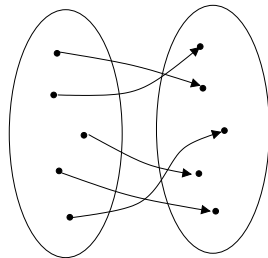⋮
(Somehow show $x_1 = x_2$.)
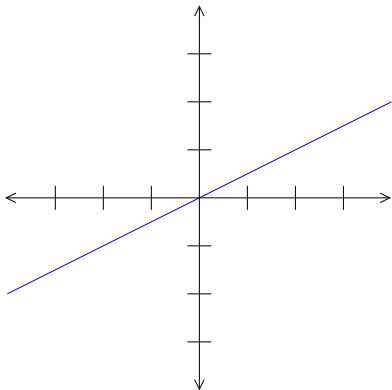Therefore f is onto.

Onto
(not one-to-one)
$|X| \geq |Y|$

One-to-one
(not onto)
$|X| \leq |Y|$

Both onto and one-to-one

$|X| = |Y|$
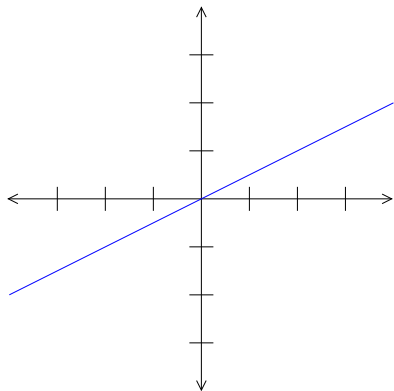
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.
Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

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Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

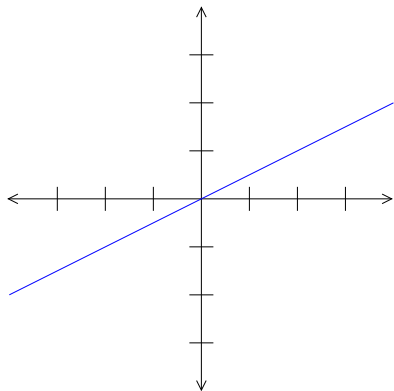$$\frac{x_1}{2} = \frac{x_2}{2}$$
$$x_1 = x_2$$

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

$$\begin{array}{rcl} \frac{x_1}{2} & = & \frac{x_2}{2} \\ x_1 & = & x_2 \end{array}$$

Therefore f is one-to-one by definition. $\square$

f is onto.

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?
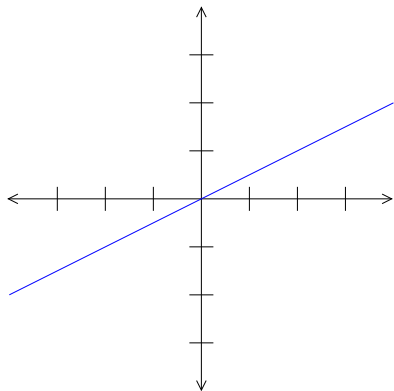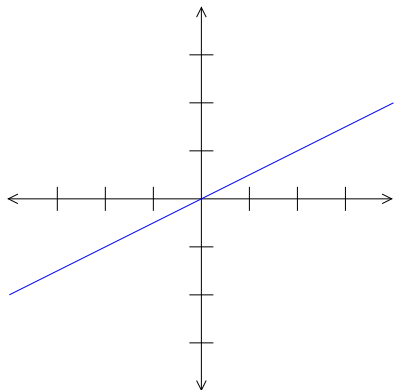
f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

$$\begin{aligned} \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Therefore f is one-to-one by definition. $\square$

f is onto.

Proof. Suppose $y \in \mathbb{R}$. *[Want x such that $f(x) = y$.]*

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

$$\frac{x_1}{2} = \frac{x_2}{2}$$
$$x_1 = x_2$$

Therefore f is one-to-one by definition. $\square$

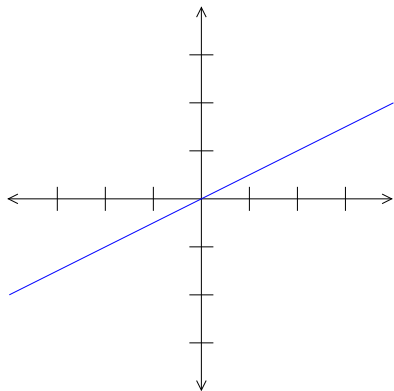f is onto.

Proof. Suppose $y \in \mathbb{R}$. *[Want x such that $f(x) = y$.]*
Let $x = 2y$. Then

$$f(x) = \frac{2y}{2}$$
$$= y$$

Therefore f is onto by definition $\square$

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Is f one-to-one? Is it onto?
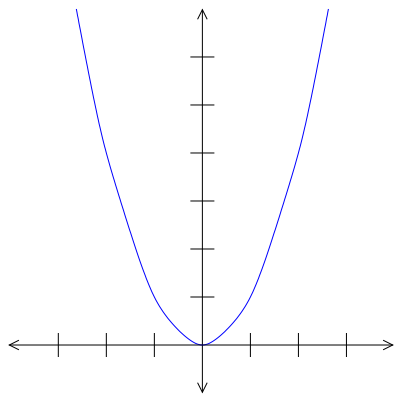
f is not one-to-one.
$f(2) = 2^2 = 4$
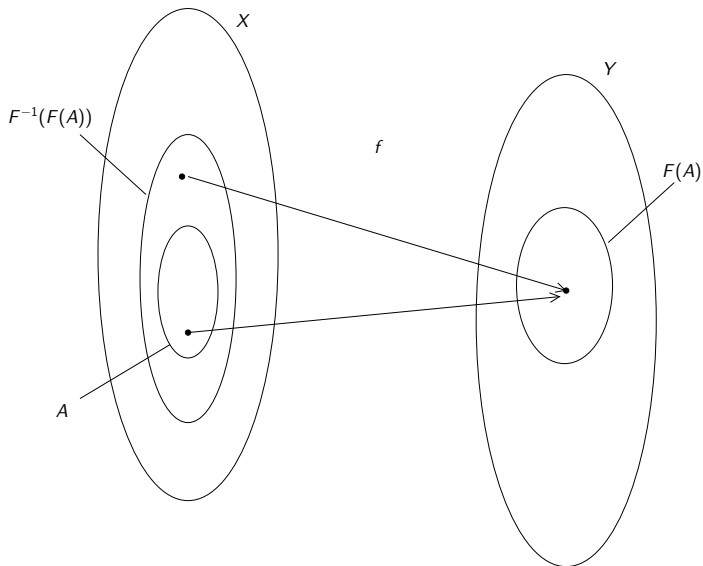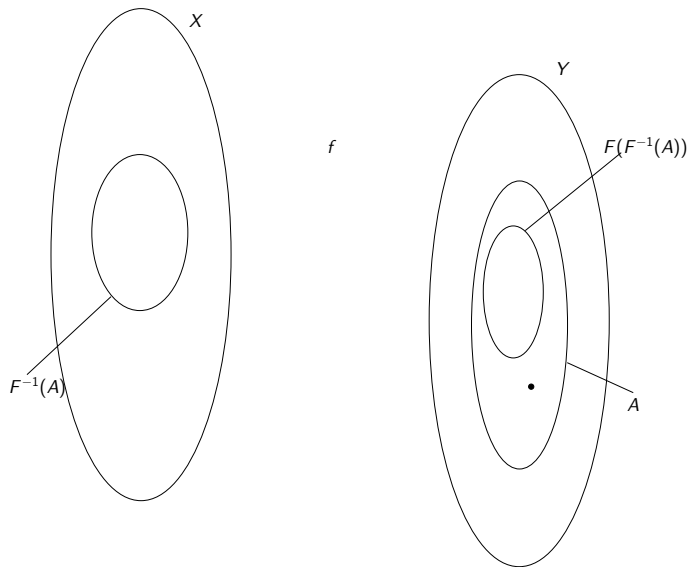$f(-2) = (-2)^2 = 4$

f is no onto.
Let $y = -1$.
$\nexists \; x \in \mathbb{R}$ such that $f(x) = -1$.

Ex 7.6.4. If $A \subseteq X$ and f is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

(Ex 7.4.9 was, Prove $A \subseteq F^{-1}(F(A))$, and Ex 7.4.10 was, Find a counterexample for $A = F^{-1}(F(A))$.)

Ex 7.6.5. If $A \subseteq Y$ and f is onto, then $A \subseteq F(F^{-1}(A))$.

Inverse relation: $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$

*Since a function is a relation, a function has an inverse, but we don't know that the inverse of a function **is a function**.*

If $f : X \to Y$ is a **one-to-one correspondence,** then

$$f^{-1} : Y \to X = \{(y, x) \in Y \times X \mid f(x) = y\}$$

is the *inverse function* of f.

Theorem 7.8 *If $f : X \to Y$ is a one-to-one correspondence, then $f^{-1} : Y \to X$ is well defined.*

Proof. Suppose $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Hence $(y, x) \in f^{-1}$ or $f^{-1}(y) = x$.

Further suppose $(y, x_1), (y, x_2) \in f^{-1}$ (*That is, suppose that both $f^{-1}(y) = x_1$ and $f^{-1}(y) = x_2$.*) Then $f(x_1) = y$ and $f(x_2) = y$. Since f is one-to-one, $x_1 = x_2$.

Therefore, by definition of function, f^{-1} is well defined. $\square$

Relation composition: If R is a relation from X to Y and S is a relation from Y to Z, then $S \circ R$ is the relation from X to Z defined as

$$S \circ R = \{(x,z) \in X \times Z \mid \exists\, y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$$

Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

$$g \circ f = \{(x,z) \in X \times Z \mid z = g(f(x))\}$$

Theorem 7.9 *If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f : X \to Z$ is well defined.*

    Proof. Suppose $x \in X$. Since f is a function, there exists a $y \in Y$ such that $f(x) = y$. Since g is a function, there exists a $z \in Z$ such that $g(y) = z$. By definition of composition, $(x,z) \in g \circ f$, or $g \circ f(x) = z$.

    Next suppose $(x, z_1), (x, z_2) \in g \circ f$, or $g \circ f(x) = z_1$ and $g \circ f(x) = z_2$. By definition of composition, there exist y_1, y_2 such that $f(x) = y_1$, $f(x) = y_2$, $g(y_1) = z_1$, and $g(y_2) = z_2$. Since f is a function, $y_1 = y_2$. Since g is a function, $z_1 = z_2$.

    Therefore, by definition of function, $g \circ f$ is well defined. $\square$

Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

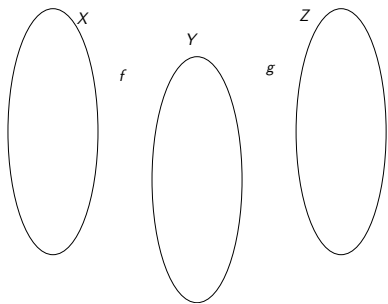$$g \circ f = \{(x,z) \in X \times Z \mid x = g(f(x))\}$$
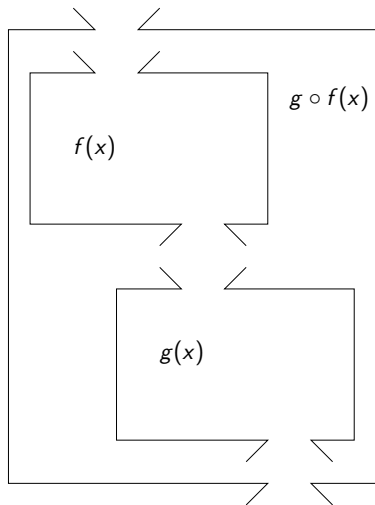
Let $f(x) = 3x$

Let $g(x) = x + 7$

Then

$$\begin{aligned} g \circ f(x) &= f(x) + 7 \\ &= 3x + 7 \end{aligned}$$

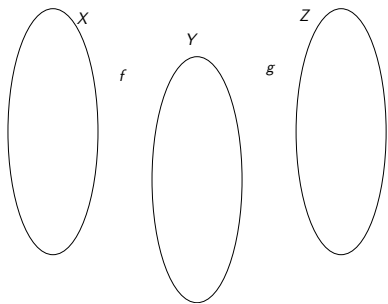**Ex 7.8.4.** If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.
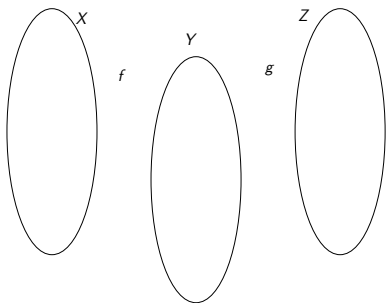
Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove "ontoness." Of which function?

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness?

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

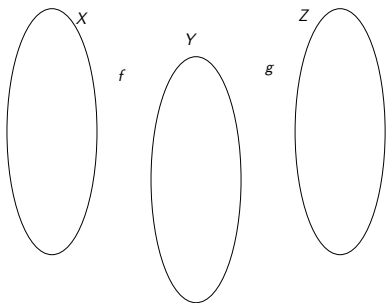*[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we're proving to be onto and show that it is hit. What is the codomain of $g \circ f$?*

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we're proving to be onto and show that it is hit. What is the codomain of $g \circ f$? Z.]
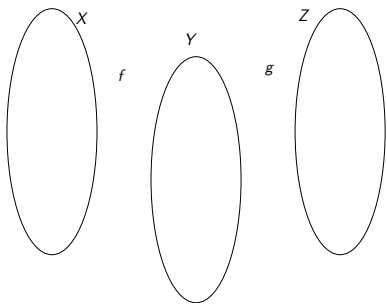
Further suppose $z \in Z$. *[We need to come up with something in the domain of $g \circ f$ that hits z. The domain is X. We will use the fact that f and g are both onto.]*

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we're proving to be onto and show that it is hit. What is the codomain of $g \circ f$? Z.]

Further suppose $z \in Z$. *[We need to come up with something in the domain of $g \circ f$ that hits z. The domain is X. We will use the fact that f and g are both onto.]*
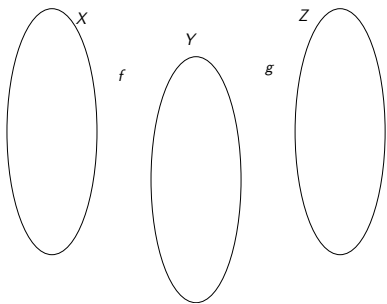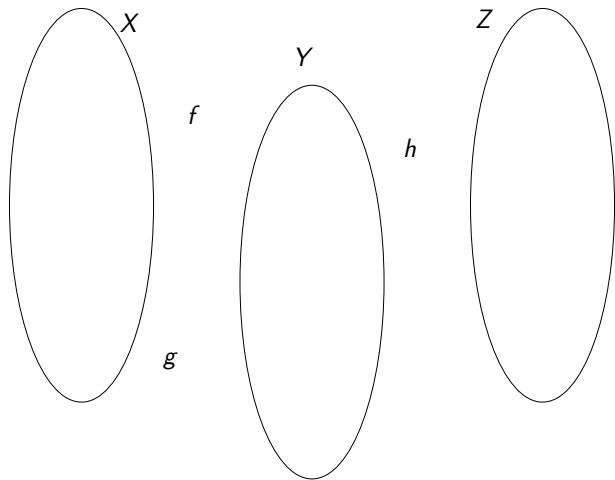

By definition of onto, there exists $y \in Y$ such that $g(y) = z$. Similarly there exists $x \in X$ such that $f(x) = y$. Now,

$$
\begin{aligned}
g \circ f(x) &= g(f(x)) \quad \text{by definition of function compos} \\
&= g(y) \quad \text{by substitution} \\
&= z \quad \text{by substitution}
\end{aligned}
$$

Therefore $g \circ f$ is onto by definition. $\square$

Ex 7.8.5. If $f : X \to Y$, $g : X \to Y$ and $h : Y \to Z$, h is one-to-one, and $h \circ f = h \circ g$, then $f = g$.

For next time:

Pg 346: 7.6.(2, 3, 6)

Ex "7.5.(a-c)" on Schoology

Pg 351: 7.8.(1, 5, 6)

Skim 7.9