Chapter 6 roadmap:

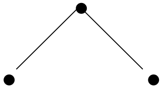
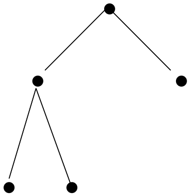
- Recursive definitions and types (Monday)
- Structural induction (**Today**)
- Mathematical induction (Friday)
- Loop invariant proofs (next week Monday and Wednesday)

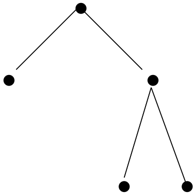
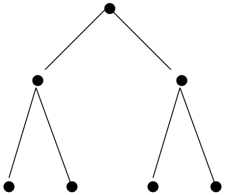
Last time we saw

- A recursive definition of whole numbers
- A recursive definition of trees, particularly *full binary trees*; a full binary tree is either
  - a leaf, or
  - an internal node together with two children which are full binary trees.

Today we see

- Self-referential proofs

Tree	Nodes	Links
	1	0
	3	2
	5	4

Tree	Nodes	Links
	5	4
	7	6

While building bigger trees from smaller trees, *the number of nodes is (and remains) one more than the number of links*. (Invariant)

Theorem 6.1 *For any full binary tree T,* $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Let $\mathcal{T}$ be the set of full binary trees. Then, we're saying

$$\forall\ T \in \mathcal{T}, \texttt{nodes}(T) = \texttt{links}(T) + 1$$

Theorem 6.1 *For any full binary tree T, $\texttt{nodes}(T) = \texttt{links}(T) + 1$.*

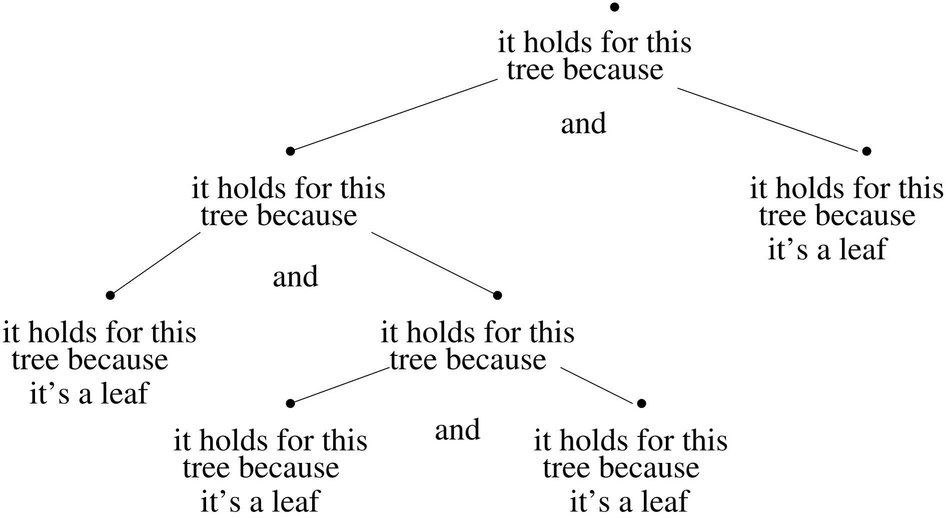
Proof. Suppose $T \in \mathcal{T}$. *[What is a tree? the definition says it's either a leaf or an internal with two subtrees. We can use division into cases.]*

Case 1. Suppose T is a leaf. Then, by how nodes and links are defined, $\texttt{nodes}(T) = 1$ and $\texttt{links}(T) = 0$. Hence $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Case 2. Suppose T is an internal node with links to subtrees T_1 and T_2. Moreover, by how nodes and links are defined, $\texttt{links}(T) = \texttt{links}(T_1) + \texttt{links}(T_2) + 2$. Then,

$$\begin{aligned}
\texttt{nodes}(T) &= 1 + \texttt{nodes}(T_1) + \texttt{nodes}(T_2) && \text{by the definition of } \texttt{nodes} \\
&= 1 + \texttt{links}(T_1) + 1 + \texttt{links}(T_2) + 1 && \text{by Theorem 6.1} \\
&= \texttt{links}(T_1) + \texttt{links}(T_2) + 2 + 1 && \text{by algebra} \\
&= \texttt{links}(T) + 1 && \text{by substitution}
\end{aligned}$$

Either way, $\texttt{nodes}(T) = \texttt{links}(T) + 1$. □


it holds for this tree because
and
it holds for this tree because
it holds for this tree because it's a leaf
and
it holds for this tree because it's a leaf
it holds for this tree because
it holds for this tree because it's a leaf
and
it holds for this tree because it's a leaf

Theorem 6.1 *For any full binary tree* T, $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Proof. Suppose $T \in \mathcal{T}$.

Base case. Suppose T is a leaf. Then, by how nodes and links are defined, $\texttt{nodes}(T) = 1$ and $\texttt{links}(T) = 0$. Hence $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Inductive case Suppose T is an internal node with links to subtrees T_1 and T_2 such that $\texttt{nodes}(T_1) = \texttt{links}(T_1) + 1$ and $\texttt{nodes}(T_2) = \texttt{links}(T_2) + 1$. Moreover, by how nodes and links are defined, $\texttt{links}(T) = \texttt{links}(T_1) + \texttt{links}(T_2) + 2$. Then,

$$
\begin{aligned}
\texttt{nodes}(T) &= 1 + \texttt{nodes}(T_1) + \texttt{nodes}(T_2) && \text{by the definition of } \texttt{nodes} \\
&= 1 + \texttt{links}(T_1) + 1 + \texttt{links}(T_2) + 1 && \text{by the inductive hypothesis} \\
&= \texttt{links}(T_1) + \texttt{links}(T_2) + 2 + 1 && \text{by algebra} \\
&= \texttt{links}(T) + 1 && \text{by substitution}
\end{aligned}
$$

Either way, $\texttt{nodes}(T) = \texttt{links}(T) + 1$. □

Let X be a a recursively defined set, and let $\{Y, Z\}$ be a partition of X, where Y is defined by a simple set of elements $Y = \{y_1, y_2, \ldots\}$ and Z is defined by a recursive rule.

Examples:

- X is the let of lists, $Y = \{[]\}$, and $Z = \{\texttt{a} :: \texttt{rest} \mid \texttt{rest} \in X\}$
- $X = \mathbb{W}$, $Y = \{0\}$, and $Z = \{\texttt{succ}(n) \mid n \in \mathbb{W}\}$
- $X = \mathcal{T}$, Y is the set of leaves, and Z is the set of internals with children $T_1, T_2 \in \mathcal{T}$.

Let X be a a recursively defined set, and let $\{Y, Z\}$ be a partition of X, where Y is defined by a simple set of elements $Y = \{y_1, y_2, \ldots\}$ and Z is defined by a recursive rule.

To prove something in the form of $\forall\, x \in X, I(x)$, do this:

Base case: Suppose $x \in Y$.

$\vdots$

$I(x)$

Inductive case: Suppose $x \in Z$. *[Using x and the definition of Z, find components $a, b, \ldots \in X$.]*

Suppose $I(a), I(b), \ldots$. *[The* **inductive hypothesis***]*

$\vdots$

Use the inductive hypothesis

$\vdots$

$I(x)$ $\square$

1. For all $x \in \mathbb{W}, 0 \leq x$
2. For all $x, y \in \mathbb{W}$, if $x \leq y$ then $succ(x) \leq succ(y)$.
3. For all $x \in \mathbb{W}, x + 0 = x$ and $0 + x = x$.
4. For all $x, y \in \mathbb{W}$, $x + succ(y) = succ(x) + y$

Exercise 6.4.5. For all $x \in \mathbb{W}$, $w \leq succ(w)$.

For next time:

Pg 268: 6.4.(3,4,6,7)

Skim 6.(5 & 6)