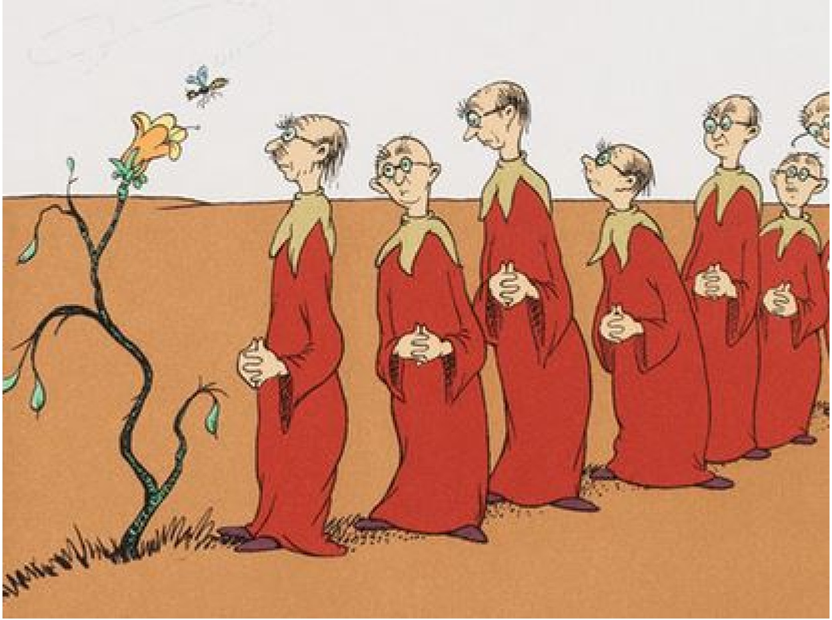
If $f : X \to Y$, $A \subseteq Y$, and f is onto, then $A \subseteq F(F^{-1}(A))$.

If $f : A \to B$ and $g : B \to C$ are both one-to-one, then $g \circ f : A \to C$ is one-to-one.

Consider the following excerpt from *Have I Ever Told You How Lucky You Are?* by Dr Seuss.

Out west, near Hawtch-Hawtch,
there's a Hawtch-Hawtcher Bee-Watcher
His job is to watch...
is to keep both his eyes on the lazy town bee.
A bee that is watched will work harder, you see.
Well... he watched and he watched.
But, in spite of his watch,
that bee didn't work any harder. Not mawtch.
So then somebody said,
"Our old bee-watching man
just isn't bee-watching as hard as he can.
He ought to be watched by *another*
Hawtch-Hawtcher.
The thing that we need
is a Bee-Watcher-Watcher."

WELL...
The Bee-Watcher-Watcher watched the
Bee-Watcher.
He didn't watch well. So another
Hawtch-Hawtcher
had to come in as a Watch-Watcher-Watcher.
And today all the Hawtchers who live in
Hawtch-Hawtch
are watching on
Watch-Watcher-Watchering-Watch,
Watch-Watching the Watcher who's watching
that bee.
You're not a Hawtch-Watcher. You're lucky, you
see.



The set of citizens of Hawtch-Hawtch, together with their bee, can be defined as the set containing the bee and all those who are employed to watch someone else in the set. We can model this using the following ML datatype:

```
datatype hawtchHawtcher = Bee | WatcherOf of hawtchHawtcher;
```

For example, the person watching the town bee is `WatcherOf(Bee)`, and the person watching the watcher of the bee-watcher is `WatcherOf(WatcherOf(WatcherOf(Bee)))`.

a. Write a function `numWatchers` that takes a hawtchHawtcher and returns the number of watchers in the chain. For example, `numWatchers(WatcherOf(WatcherOf(WatcherOf(WatcherOf(Bee)))))` would return 4.

b. Write a function `beeProductivity` that takes a hawtchHawtcher and two reals indicating the pollenation rate of the bee if it were unwatched and the factor by which the bee's pollentation improves for each watcher in the chain watching it. For exampe,
`beeProductivity(WatcherOf(WatcherOf(Bee)), 0.3, 1.25)`
returns `0.46875` which is `1.25 * 1.25 * 0.3`.

c. Write a function `doubleWatchers` that takes a hawtchHawtcher and returns a hawtchHawtcher like the given one except with twice as many watchers. For example, `doubleWatchers(WatcherOf(Bee))` would return `WatcherOf(WatcherOf(Bee))`.

Prove that $I(n)$ is a loop invariant for `bbb`. (14 points.)

$$I(n) = \quad \text{after } n \text{ iterations, } x = 50 + i$$

```
fun bbb(m) =
  let
    val x = ref 50;
    val y = ref 50;
    val i = ref 0;
  in
   (while !i < m do
     (x := !x + 1;
      y := !y - 1;
      i := !i + 1);
    !x + !y)
  end;
```

Write a function `findExtreme` that takes a function (with type int $\times$ int $\rightarrow$ bool) and a list of integers and uses the function to select the extreme element (least, greatest, etc) of the list. Specifically, the function that `findExtreme` takes as a parameter defines a way to order int, that is, it compares two ints (say *a* and *b*) and returns `true` if *a* comes before *b* and `false` otherwise (mathematically, this function is a *total order*). Thus `findExtreme` is a generalization of `findGreatest`. For example, `findExtreme(fn (a, b) => a > b, [6, 4, 18, 9, 2])` would return `18`. (This problem is *not* naturally solved using `map` or `filter`.)