

Chapter 7 outline:

- ▶ Introduction, function equality, and anonymous functions (Friday, Nov 11)
- ▶ Image and inverse images (last week Monday)
- ▶ Function properties, composition, and applications to programming (last week Wednesday)
- ▶ Cardinality (last week Friday)
- ▶ *Practice quiz* and Countability (**Today**)
- ▶ Review (Monday, Nov 28)
- ▶ Test 3, on Ch 6 & 7 (Wednesday, Nov 30)

Two finite sets X and Y have the *the same cardinality* as each other if there exists a one-to-one correspondence from X to Y .

To use this *analytically*:

Suppose X and Y have the same cardinality. Then let f be a one-to-one correspondence from X to Y .

f is both onto and one-to-one.

To use this *synthetically*:

Given sets X and $Y \dots$

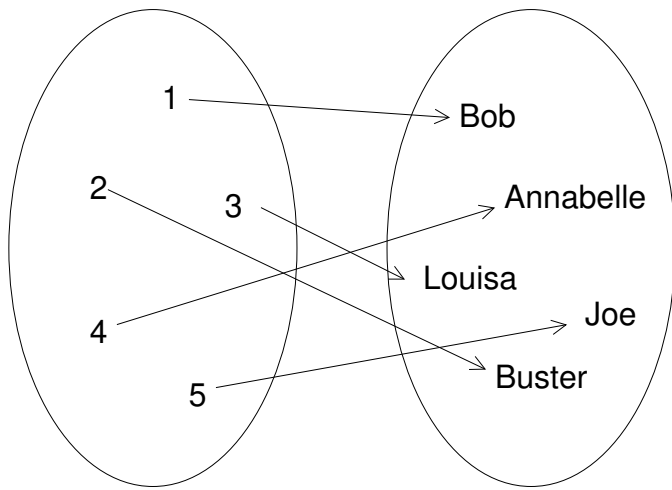
[Define f] Let $f : X \rightarrow Y$ be a function defined as \dots

Suppose $y \in Y$. *Somehow find* $x \in X$ such that $f(x) = y$. Hence f is onto.

Suppose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. *Somehow show* $x_1 = x_2$. Hence f is one-to-one.

Since f is a one-to-one correspondence, X and Y have the same cardinality as each other.

A finite set X has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \dots, n\}$ to X . Moreover, $|\emptyset| = 0$.



Two finite sets X and Y have the *the same cardinality* as each other if there exists a one-to-one correspondence from X to Y .

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Given a set X , if there exists $n \in \mathbb{N}$ and a one-to-one correspondence from $\{1, 2, \dots, n\}$ to X , then X is *finite* and $|X| = n$. Otherwise, X is *infinite*.

Are all infinities equal?

Which is more intuitive to you,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{R}|$$

or

$$|\mathbb{N}| < |\mathbb{W}| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|$$

?

Thm 7.19. \mathbb{W} and \mathbb{N} have the same cardinality.

Proof. [We need a one-to-one correspondence from \mathbb{N} to \mathbb{W} .]

Let $f : \mathbb{W} \rightarrow \mathbb{N}$ be defined so that $f(n) = n + 1$.

Suppose $n \in \mathbb{N}$. Then $f(n - 1) = (n - 1) + 1 = n$, so f is onto.

Next suppose $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$, and moreover $n_1 = n_2$. Hence f is one-to-one.

Since a one-to-one correspondence exists between \mathbb{W} and \mathbb{N} , the two sets have the same cardinality. \square

A set X is *countably infinite* if there exists a one-to-one correspondence from \mathbb{N} to X .

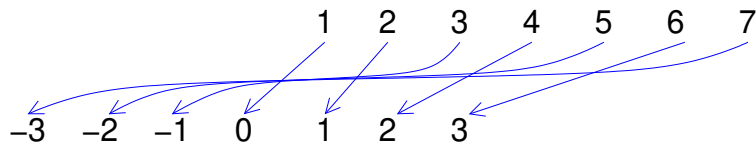
A set is *countable* if it is finite or countably infinite. Otherwise it is *uncountable*.

Thm 7.20. \mathbb{Z} is countably infinite.

Proof. [We need a one-to-one correspondence from \mathbb{N} to \mathbb{Z} .]

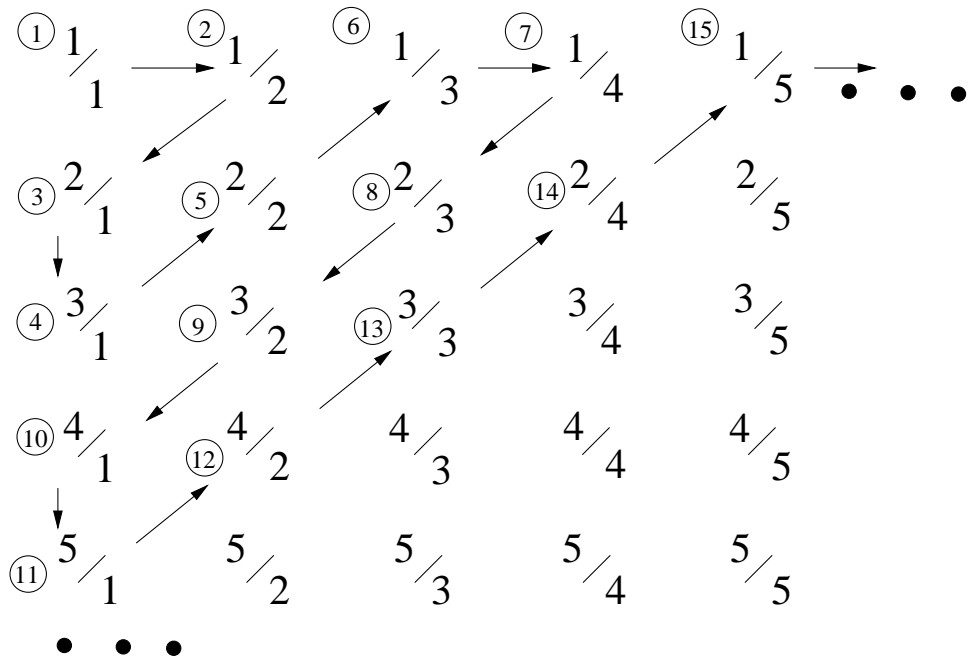
Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be defined so that

$$f(x) = \begin{cases} n \operatorname{div} 2 & \text{if } n \text{ is even} \\ -(n \operatorname{div} 2) & \text{otherwise} \end{cases}$$



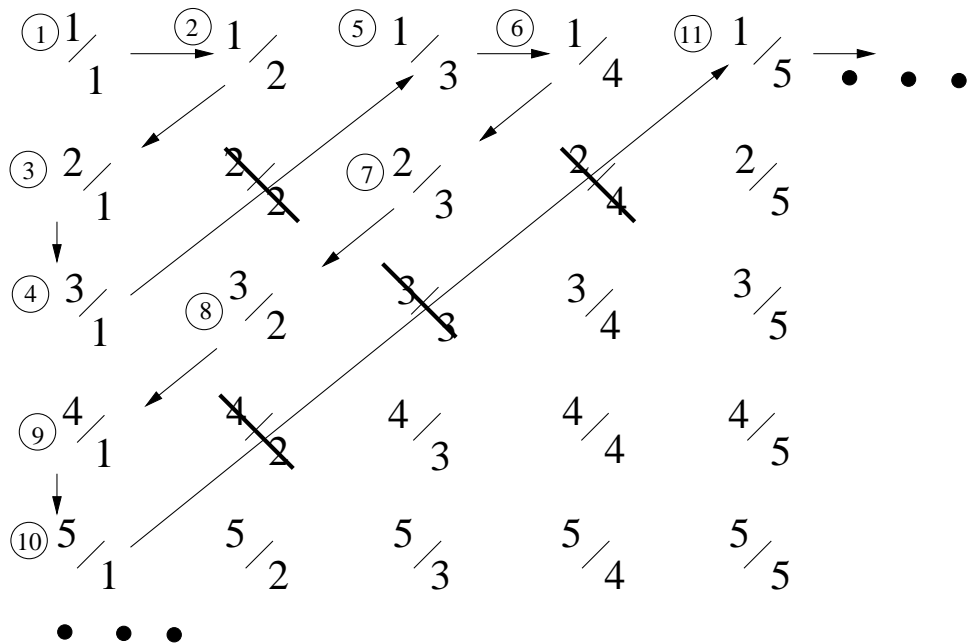
Since f is a one-to-one correspondence, \mathbb{Z} is countably infinite. \square

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	• • •
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	
$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	
• • •					



```
fun findRoom(busNum, seatNum) =  
  let  
    fun nextPair(a, b) =  
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)  
      else if b = 1 andalso a mod 2 = 0  
        then (a + 1, 1)  
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)  
      else (a - 1, b + 1);  
    fun findRoomHelper(i, currentPair) =  
      if currentPair <> (busNum, seatNum)  
      then findRoomHelper(i + 1, nextPair(currentPair))  
      else i;  
  in  
    findRoomHelper(1, (1, 1))  
  end;
```

```
fun findBusSeat(room) =  
  let  
    fun nextPair(a, b) =  
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)  
      else if b = 1 andalso a mod 2 = 0  
        then (a + 1, 1)  
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)  
      else (a - 1, b + 1);  
    fun findBusSeatHelper(i, currentPair) =  
      if i <> room  
      then findBusSeatHelper(i + 1,  
                             nextPair(currentPair))  
      else currentPair;  
  in  
    findBusSeatHelper(1, (1, 1))  
  end;
```



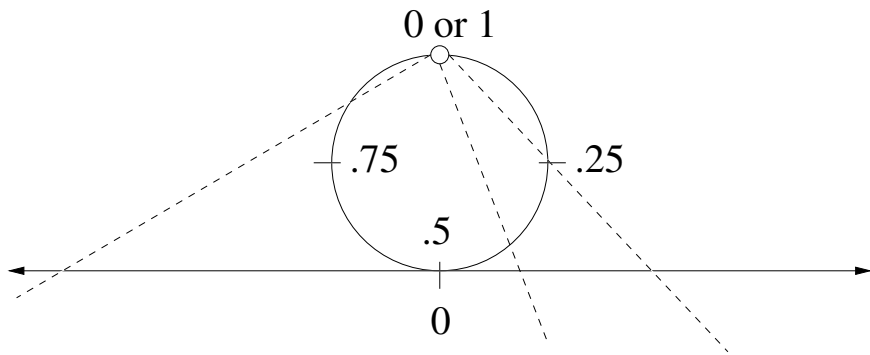
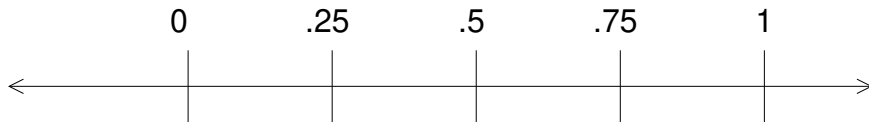
Thm 7.21. \mathbb{Q}^+ is countably infinite.

So,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}|$$

What about \mathbb{R} ?

Thm 7.22. $(0, 1)$ has the same cardinality as \mathbb{R} .



Thm 7.23. $(0, 1)$ is uncountable.

Proof. Suppose $(0, 1)$ is countable. Then there exists a one-to-one correspondence $f : \mathbb{N} \rightarrow (0, 1)$. We will use f to give names to all the digits of all the numbers in $(0, 1)$, considering each number in its decimal expansion, where each $a_{i,j}$ stands for a digit.:

$$\begin{aligned} f(1) &= 0.a_{1,1}a_{1,2}a_{1,3}\dots a_{1,j}\dots \\ f(2) &= 0.a_{2,1}a_{2,2}a_{2,3}\dots a_{2,j}\dots \\ &\vdots \\ f(x) &= 0.a_{x,1}a_{x,2}a_{x,3}\dots a_{x,j}\dots \\ &\vdots \end{aligned}$$

Now construct a number $d = 0.d_1d_2d_3\dots d_i\dots$ as follows

$$d_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1 \\ 2 & \text{if } a_{i,i} = 1 \end{cases}$$

Since $d \in (0, 1)$ and f is onto, there exists an $x \in \mathbb{N}$ such that $f(x) = d$.
Moreover,

$$f(x) = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

so

$$d = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

by substitution. In other words, $d_i = a_{x,i}$, and specifically $d_x = a_{x,x}$. However, by the way that we have defined d , we know that $d_x \neq a_{x,x}$, a contradiction. Therefore $(0, 1)$ is not countable. \square