Chapter 7 in context:

- Chapter 5 Relations: Builds on proofs about sets
- Chapter 6 Self Reference: Interlude between Chapters 5 and 7, focuses on recursive thinking
- Chapter 7 Function: Builds on proofs about relations

Chapter 7 outline:

- Introduction, function equality, and anonymous functions (**Today**)
- Image and inverse images (next week Monday)
- Function properties, composition, and applications to programming (next week Wednesday)
- Cardinality (next week Friday)
- Countability (Monday before Thanksgiving)
- Review (Monday after Thanksgiving)
- Test 3, on Ch 6 & 7 (Wednesday, Nov 30)

Cross out the term/concept that was **not** used in the reading for today as a way to think about functions

A kind of machine

A form of induction

A mapping between two collections

A kind of relation

For the function $f : X \to Y$, X is the ____________________ and Y is the ____________________.

function

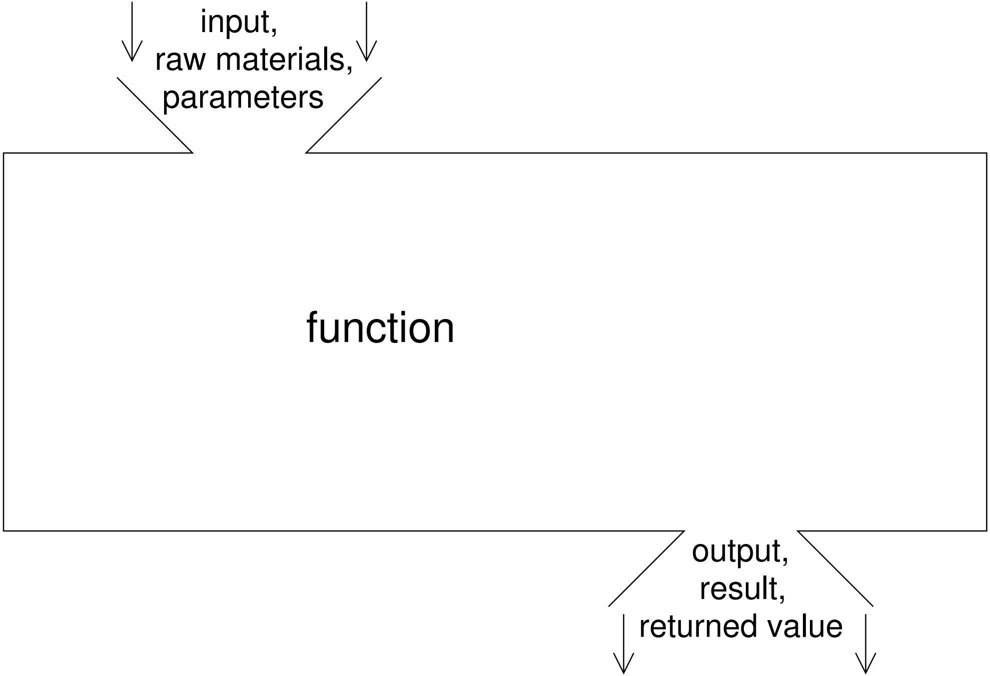
constant

domain

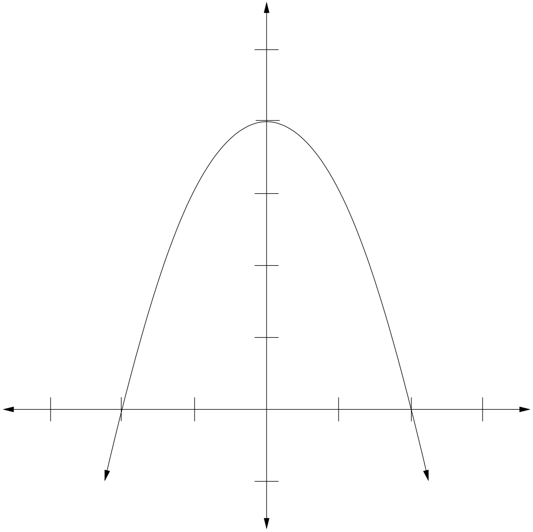
codomain

first-class value

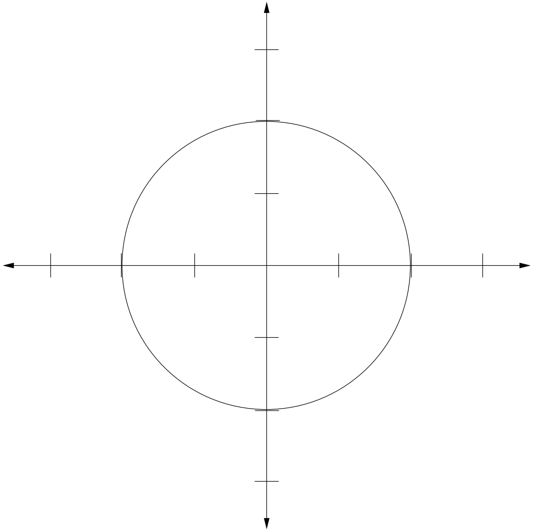
relation


input,
raw materials,
parameters
function
output,
result,
returned value

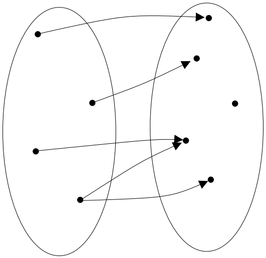
Alice	x3498
Bob	x4472
Carol	x5392
Dave	x9955
Eve	x2533
Fred	x9448
Georgia	x3684
Herb	x8401



$y = 4 - x^2$

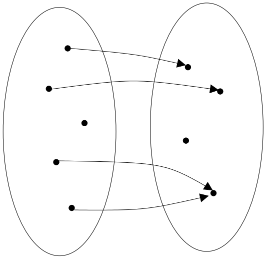


$y^2 = 4 - x^2$



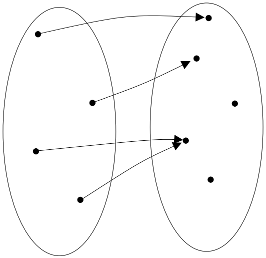
Not a function.

(There's a domain element that is related to two things.)



Not a function.

(There's a domain element that is not related to anything.)



A function.

(It's OK that two domain elements are related to the same thing and one codomain element has nothing related to it.)

Definition of function

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal: $f \subseteq X \times Y$ is a *function* if

$$\begin{array}{lll} \forall\, x \in X, & \exists\, y \in Y \mid (x, y) \in f & \textbf{existence} \text{ of } y \\ & \wedge \quad \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \to y_1 = y_2 & \textbf{uniqueness} \text{ of } y \end{array}$$

Change of notation

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal (relation notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \quad \exists\, y \in Y \mid (x, y) \in f \qquad \textbf{existence of } y$$

$$\wedge \quad \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2 \qquad \textbf{uniqueness of } y$$

Formal (function notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \quad \exists\, y \in Y \mid f(x) = y \qquad \textbf{existence of } y$$

$$\wedge \quad \forall\, y_1, y_2 \in Y, (f(x) = y_1 \wedge f(x) = y_2) \rightarrow y_1 = y_2 \qquad \textbf{uniqueness of } y$$

We call X the *domain* and Y the *codomain* of f.

Definition of function equality. Let $f, g : X \to Y$

Old definition: functions are sets.

$$f = g \text{ if } \forall\ f \subseteq g \ \wedge g \subseteq f$$

New definition: based on function notation.

$$f = g \quad \text{if} \ \ \forall\ x \in X, f(x) = g(x)$$

Function equality: $f = g$ if $\forall\ x \in X, f(x) = g(x)$

Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x \cdot (x - 1) - 6$ and $g(x) = (x - 3)(x + 2)$.

Prove $f = g$.

The old and new definitions of function equality are equivalent.

Ex 7.2.1. $(\forall\, x \in X, f(x) = g(x)) \quad \text{iff} \quad (f \subseteq g \wedge g \subseteq f)$.

The old and new definitions of function equality are equivalent.

Ex 7.2.1. $(\forall\, x \in X, f(x) = g(x))$ iff $(f \subseteq g \wedge g \subseteq f)$.

Proof. First, suppose $\forall\, x \in X, f(x) = g(x)$, that is, $f = g$ by definition of function equality. Further suppose $(x, y) \in f$. By function notation, $f(x) = y$. By supposition and substitution, $g(x) = y$. By relation notation, $(x, y) \in g$. Finally, $f \subseteq g$ by definition of subset.

Similarly $g \subseteq f$, and therefore $f = g$ by definition of set equality.

Conversely, suppose $f \subseteq g \wedge g \subseteq f$, that is, $f = g$ by definition of set equality. Further suppose $x \in X$.

Let $y = f(x)$. Note that this $y \in Y$ must exist by definition of function. By relation notation, $(x, y) \in f$.

By definition of subset [or set equality], $(x, y) \in g$. In function notation, that is $g(x) = y$, and so $f(x) = g(x)$ by substitution. Therefore $f = g$ by definition of function equality. $\square$

For next time:

Pg 331: 7.2.(2 & 3)
Pg 335: 7.3.(3, 4, 8)

Read 7.4
Skim 7.5