Proof of Horner's rule loop invariant $\left(y = \sum_{k=0}^{n-(i+1)} a_{k+i+1}x^k\right)$:

Init. *After 0 iterations, $y = 0$, $i = n$ by assignment. So*

$$\sum_{k=0}^{n-(i+1)} a_{k+i+1} = \sum_{k=0}^{-1} a_{k+i+1}x^k = 0 = y$$

Maint. *Now, suppose this holds true after N iterations, that is*

$$y_{old} = \sum_{k=0}^{n-(i_{old}+1)} a_{k+i_{old}+1}x^k$$

where y_{old} and i_{old} are y and i after N iterations. Likewise, let y_{new} and i_{new} be the values after $N + 1$ iterations.

By assignment $i_{new} = i_{old} - 1$. *Then*

$$
\begin{aligned}
y_{new} &= a_{i_{old}} + x \cdot y_{old} && \text{by assignment} \\
&= a_{i_{old}} + x \cdot \sum\nolimits_{k=0}^{n-(i_{old}+1)} a_{k+i_{old}+1} x^k && \\
&= a_{i_{new}-1} + x \cdot \sum\nolimits_{k=0}^{n-(i_{new}+2)} a_{k+i_{new}} x^k && \text{by substitution} \\
&= a_{i_{new}-1} + \sum\nolimits_{k=0}^{n-(i_{new}+2)} a_{k+i_{new}} x^{k+1} && \text{by distribution} \\
&= a_{i_{new}-1} + \sum\nolimits_{k=1}^{n-(i_{new}+1)} a_{k+i_{new}+1} x^k && \text{by change of variables} \\
&= a_{0+i_{new}-1} x^0 + \sum\nolimits_{k=1}^{n-(i_{new}+1)} a_{k+i_{new}+1} x^k && \\
&= \sum\nolimits_{k=0}^{n-(i_{new}+1)} a_{k+i_{new}+1} x^k && \square
\end{aligned}
$$

Formal definition of big-Theta:

$$\Theta(g(n)) = \{f(n) \mid \exists\ c_1, c_2, n_0 \in \mathbb{N} \text{ such that } \forall\ n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

$g(n) = \frac{1}{2}n^2 - 3n = \Theta(n^2)$.

Proof. *Let* $c_1 = \frac{1}{14}$, $c_2 = \frac{1}{2}$ *and* $n_0 = 7$. *Suppose* $n > 7$. *Then*

$$\begin{array}{rcccl} \frac{1}{14} & = & \frac{1}{2} - \frac{3}{7} < \frac{1}{2} & & \\ \frac{1}{14} & \leq & \frac{1}{2} - \frac{3}{n} & \leq & \frac{1}{2} \\ \frac{n^2}{14} & \leq & \frac{1}{2}n^2 - 3n & \leq & \frac{n^2}{2} \\ c_1 n^2 & \leq & g(n) & \leq & c_2 n^2 \end{array}$$

Therefore $g(n) = \Theta(n^2)$ *by definition.* $\square$

Theorem 3.1. For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Proof. *Suppose* $f = \Theta(g(n))$. *Then, by definition of* Θ, *there exist constants* c_1, c_2, *and* n_0 *such that for all* $n \geq n_0$,

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

Let $c = c_2$. *Then* $0 \leq f(n) \leq c \cdot g(n)$, *hence* $f(n) = O(g(n))$ *by definition. Similarly, let* $c = c_1$. *Then* $0 \leq c \cdot g(n)$, *hence* $f(n) = \Omega(g(n))$.

Conversely, suppose $f(n) = O(g(n))$ *and* $f(n) = \Omega(g(n))$. *By the definitions, there exist* c, *and* n_1 *such that for all* $n \geq n_1$, $0 \leq f(n) \leq c \cdot g(n)$, *and there exist* c', *and* n_1' *such that for all* $n \geq n_1'$, $0 \leq c' \cdot g(n) \leq f(n)$.
Let $c_1 = c'$, $c_2 = c$, *and* $n_0 = max(n_1, n_1')$. *Hence* $f(n) = \Theta(g(n))$. $\square$

3.1-4. Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

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To see that $2^{n+1} = O(2^n)$, note that $2^{n+1} = 2 \cdot 2^n$. Thus 2 is the constant we're looking for, and we're done.

Let's attempt a proof that $2^{2n} = O(2^n)$. Does $\exists\ c, n_0 \mid \forall\ n \leq n_0, 2^{2n} \leq c \cdot 2^n$? If so, then

$$\begin{array}{rcl} 2^n \cdot 2^n & \leq & c \cdot 2^n \\ 2^n & \leq & c \end{array}$$

...which is impossible.

3-1.d. If $k > d$, then $p(n) = o(n^k)$.

Proof. Suppose $k > d$ and suppose $c > 0$. Then

$$\begin{array}{rcll} a_0 + a_1 n + \ldots + a_d n^d & < & a_x + a_x n + \ldots + a_x n^d & \text{where } a_x = \max(a_0, a_1, \ldots a_d) \\ & < & d \cdot a_x n^d & \text{(see why I chose } a_x \text{ instead of } a_m\text{?)} \\ & < & c \cdot n^k & \text{if } n \text{ is big enough.} \end{array}$$

So, we want $d \cdot a_x < c \cdot n^{k-d}$. This holds as long as

$$n > \left(\frac{d \cdot a_x}{c} \right)^{\frac{1}{k-d}}$$