If $f(n) = \omega(g(n))$ then $f(n) \neq O(g(n))$.

Proof. *Suppose $f(n) = \omega(g(n))$. Further suppose $f(n) = O(g(n))$.*

By definition of big-Oh, there exists c and n_0 such that for all $n \geq n_0$, $0 \leq f(n) \leq cg(n)$. By definition of little-omega, there exists n_1 such that for all $n \geq n_1$, $cg(n) < f(n)$.

Let $n_2 = \max(n_0, n_1)$. Then $cg(n_2) < f(n) \leq cg(n_2)$, which is a contradiction.

Therefore $f(n) \neq O(g(n))$. $\square$

No algorithm to transform an arbitrary binary tree with n comparable keys to a binary search tree with the same keys in expected time $o(n \lg n)$ exists.

Proof. *Suppose such an algorithm for BST-building exists. Then, construct the following algorithm:*

1. *Given an array with of comparable keys, transform that array into a binary tree, such as by making it a long dangly list-like tree. This takes* $\Theta(n)$ *time.*
2. *Transform that binary tree into a BST using the supposed algorithm. This takes* $o(n \lg n)$ *time, according to our supposition.*
3. *Transform the BST into a sorted array by traversing the tree. This takes* $\Theta(n)$ *time.*

This algorithm sorts the given array, and takes $\Theta(n) + o(n \lg n) + \Theta(n) = o(n \lg n)$ *time. By Theorem 8.1, this is impossible. Therefore no such algorithm for BST-building exists.* □

4.c. Consider the sequence of `delete()` operations from just after a defragmentation up through and including the next defragmenting. Let m be `size` at the beginning of this sequence. There will be $\frac{m}{2} - 1$ operations that are constant time, and the, last, defragmenting operation costs $O(m)$. All together this costs $O(m)$. Using the aggregate method, we spread the $O(m)$ cost across the $\frac{m}{2}$ operations to consider them $O(1)$ each. Using the accounting method, charge each of the $\frac{m}{2} - 1$ non-defragmenting operations 3 units, one of the nulling of the position itself, and two for its contribution to the next defragmenting (one for a nulling, one for a filling).

A *field* is a set together with two binary operations satisfying certain properties, known as the *field axioms*. The real numbers with addition and multiplication is the canonical example, but everything in the FFT works for other fields (such as rational numbers and complex numbers).

A *polynomial* is a function of x in the form

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

A polynomial with *degree* k has $k+1$ indices and least *degree bound* $k+1$.

A polynomial with degree bound n has degree no greater than $n-1$ and has (at most) n coefficients.

The product of two polynomials with (least common) degree bound n, has degree bound $2n-1$.

Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and r such that $A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

$$
\begin{aligned}
c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n &= (q_0 + q_1 x + \cdots + q_{n-1}x^{n-1})(x - x_0) + r \\
&= q_0 x + q_1 x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n \\
&\quad -q_0 x_0 - q_1 x_0 x - \cdots - q_{n-1} x_0 x^{n-1} + r \\
&= (r - q_0 x_0) + (q_0 - q_1 x_0)x + \cdots \\
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Ex 30.1-4. Prove that n distinct point-value pairs are necessary to uniquely specify a polynomial of degree-bound n. (Fewer than n distinct pairs fail to specify a unique polynomial.)

Proof. *Let* $\{(x_0, y_0), \ldots (x_{n-2}, y_{n-2})\}$ *be the set of points. WOLOG, assume no* $x_i = 0$*.*

Add the point $(0, 0)$ *to the set. By Theorem 30.1, this set specifies a unique n-degree-bound function* $A(x)$*.*

Alternately, add the point $(0, 1)$ *to the set. By Theorem 30.1, this set specifies a unique n-degree-bound function* $B(x)$*.*

Since $A(0) = 0 \neq 1 = B(0)$*, it must be that* $A \neq B$*. Therefore n points are necessary.* $\square$*.*

Given n points, $(x_0, y_0), \ldots (x_{n-1}, y_{n-1})$,

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k}(x - x_j)}{\prod_{j \neq k}(x_k - x_j)}$$