Formal definitions:

$$\Theta(g(n)) = \{f(n) \mid \exists\ c_1, c_2, n_0 \in \mathbb{R}^+ \text{ such that } \forall\ n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

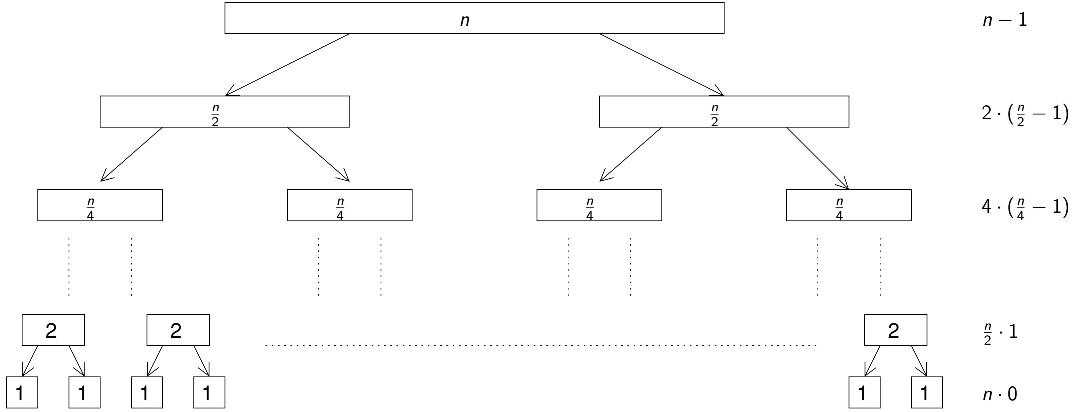
$$O(g(n)) = \{f(n) \mid \exists\ c, n_0 \in \mathbb{R}^+ \text{ such that } \forall\ n \geq n_0, 0 \leq f(n) \leq c\ g(n)\}$$

$$\Omega(g(n)) = \{f(n) \mid \exists\ c, n_0 \in \mathbb{R}^+ \text{ such that } \forall\ n \geq n_0, 0 \leq c\ g(n) \leq f(n)\ \}$$

$$o(g(n)) = \{f(n) \mid \forall\ c \in \mathbb{R}^+, \exists\ n_0 \in \mathbb{R}^+ \text{ such that } \forall\ n \geq n_0, 0 \leq f(n) < c\ g(n)\ \}$$

$$\omega(g(n)) = \{f(n) \mid \forall\ c \in \mathbb{R}^+, \exists\ n_0 \in \mathbb{R}^+ \text{ such that } \forall\ n \geq n_0, 0 \leq c\ g(n) < f(n)\ \}$$

$$C_{ms}(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ n - 1 + 2C_{ms}(\frac{n}{2}) & \text{otherwise} \end{cases}$$



The Master theorem (CLRS pg 94). Let $T : \mathbb{N} \to \mathbb{N}$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$ and $b > 1$.

- If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- If $f(n) = \Theta(n^{\log_b a})$ for some $\epsilon > 0$, then $f(n) = \Theta(n^{\log_b a} \lg n)$
- If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ and $a \cdot f\left(\frac{n}{b}\right) = O(f(n))$, then $T(n) = \Theta(f(n))$

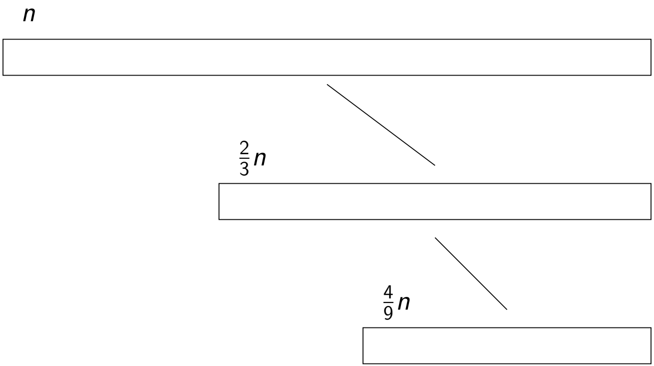
Understanding the Master theorem:

The work done at the leaves is inherently $\Theta(n^{\log_b a})$

binary search	merge sort
$a = 1 \quad b = 2$	$a = 2 \quad b = 2$
1 leaf	n leaves
$n^{\log_2 1} = n^0 = 1$	$n^{\log_2 2} = n^1$

Understanding the Master theorem:

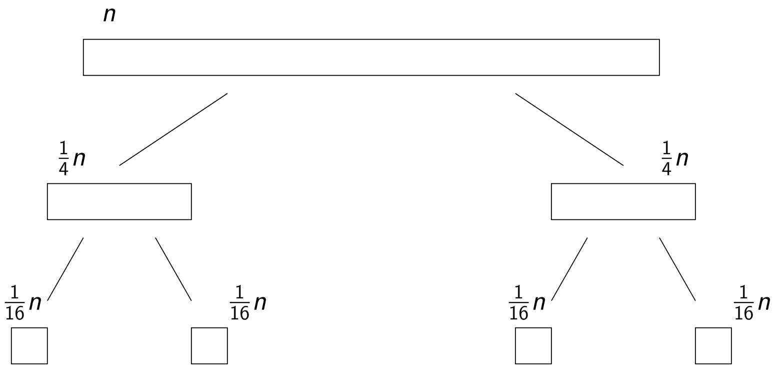
Imagine throwing away $\frac{1}{3}$ of the problem each time.



$a = 1$, $b = \frac{3}{2}$, number of leaves: $n^{\log_{\frac{3}{2}} 1} = 1$.

Understanding the Master theorem:

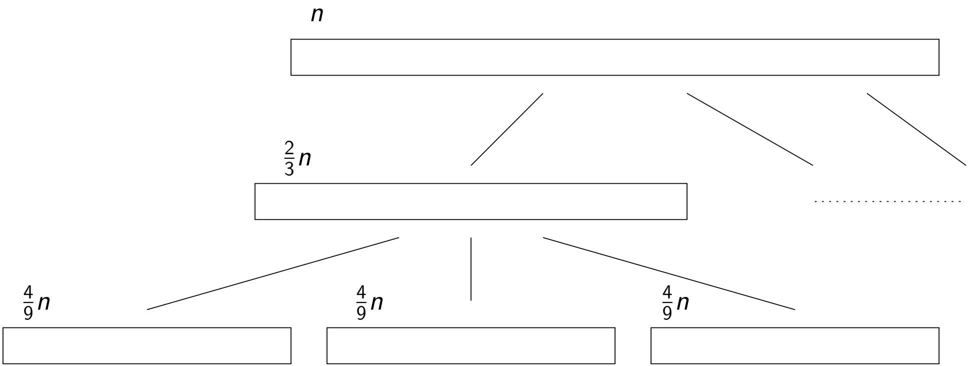
Imagine throwing away two quarters of the problem each time, keeping two (independent) quarters.



$a = 2$, $b = 4$, number of leaves: $n^{\log_4 2} = n^{\frac{1}{2}}$.

Understanding the Master theorem:

Imagine three *overlapping* subproblems, each with size $\frac{2}{3}$



$a = 3$, $b = \frac{3}{2}$, number of leaves: $n^{\log_{\frac{3}{2}} 3} \approx n^{2.7}$.

Understanding the Master theorem:

Let a be the number of subproblems and b be the factor by which the subproblems are decreasing in size (size of subproblems are $\frac{n}{b}$). Then

- The number of leaves is $\Theta(n^{\log_b a})$.
- Assuming a constant amount of work for each leaf, the work done at the leaves is $\Theta(n^{\log_b a})$.
- Thus the total work done by the algorithm is $\Omega(n^{\log_b a})$.

Understanding the Master theorem:

In the recursion tree, what dominates—the work at the *root* or at the *leaves*?

- If one clearly (*polynomially*) dominates, then the whole work is Θ of whichever it is.
- If they're asymptotically equivalent, then multiply the work at each level by the height of the tree, which is $\Theta(\lg n)$.
- Otherwise, you're out of luck.

Understanding the Master theorem—a less-formal, big-oh-only version:

If $T(n) \leq aT(\frac{n}{b}) + O(n^d)$, then

$$T(n) = \begin{cases} O(n^d \lg n) & \text{if } a = b^d \quad \text{(same work at each level)} \\ O(n^d) & \text{if } a < b^d \quad \text{(root dominates)} \\ O(n^{\lg_b a}) & \text{if a} > b^d \quad \text{(leaves dominate)} \end{cases}$$

The Master theorem (CLRS pg 94). Let $T : \mathbb{N} \to \mathbb{N}$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$ and $b > 1$.

- If $f(n) = O(n^{\log_b a-\epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- If $f(n) = \Theta(n^{\log_b a})$ for some $\epsilon > 0$, then $f(n) = \Theta(n^{\log_b a} \lg n)$
- If $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some $\epsilon > 0$ and $a \cdot f\left(\frac{n}{b}\right) = O(f(n))$, then $T(n) = \Theta(f(n))$

The "regularity" condition, that is there exists c such that

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

> *The amount of work at the next level needs to be "smaller" (asymptotically no bigger than) the work at the current level.*

Ex. 4.5-1.

In each of these, $a = 2$, $b = 4$. $n^{\log_b a} = n^{\log_4 2} = n^{\frac{1}{2}}$

a. $T(n) = 2T(\frac{n}{4}) + 1$

$$f(n) = 1 = O(n^{\log_4 2 - \epsilon}) = O(n^{\frac{1}{2} - \epsilon})$$

where $\epsilon = \frac{1}{4}$, for example. hence $\Theta(n^{\frac{1}{2}})$.

b. $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$

$$f(n) = \sqrt{n} = n^{\frac{1}{2}} = \Theta(n^{\frac{1}{2}})$$

So $\Theta(n^{\frac{1}{2}} \lg n)$

Ex. 4.5-1.

$a = 2,\ b = 4.\ n^{\log_b a} = n^{\log_4 2} = n^{\frac{1}{2}}$

c. $T(n) = 2T(\frac{n}{4}) + n$

$$f(n) = n = \Omega(n^{\frac{1}{2}+\epsilon})$$

where $\epsilon = \frac{1}{4}$, for example. So $\Theta(n)$.

d. $T(n) = 2T(\frac{n}{4}) + n^2$

$$f(n) = n^2 = \Omega(n^{\frac{1}{2}+\epsilon})$$

where $\epsilon = 1$, for example. So $\Theta(n^2)$.

4-1.

a. $T(n) = 2T(\frac{n}{2}) + n^4$

Using the master method, $a = 2$, $b = 2$, and $f(n) = n^4 = \Omega(n^{1+\epsilon})$ where $\epsilon = 2$. So, $\Theta(n^4)$.

Using the substitution method, guess cn^4. Then

$$
\begin{aligned}
T(n) &= 2T(\tfrac{n}{2}) + n^4 \\
&= 2c(\tfrac{n}{2})^4 + n^4 \\
&= \tfrac{c}{8}n^4 n^4 \\
&= (\tfrac{c+8}{8})n^4
\end{aligned}
$$

We need $\frac{c+8}{8} = c$, so $c = \frac{8}{7}$.

4-1.

b. Using the master method, $a = 1$, $b = \frac{10}{7}$. Note that $\log_{\frac{10}{7}} 1 = 0$. $f(n) = n = \Omega(n^{0+\epsilon})$ where $\epsilon = \frac{1}{2}$. So, $\Theta(n)$.

Using the substitution method, guess cn. Then

$$
\begin{aligned}
T(n) &= T(\tfrac{7n}{10}) + n \\
&= c(\tfrac{7}{10}n) + n \\
&= \tfrac{7c+10}{10}n \\
\tfrac{7c+10}{10} &= c \\
7c + 10 &= 10c \\
c &= \tfrac{10}{3}
\end{aligned}
$$

4.5-4. Can we use the Master method on $T(n) = 4T(\frac{n}{2}) + n^2 \lg n$?
No. $a = 4$, $b = 2$, $\log_2 4 = 2$. Note that $n^2 \lg n = \Omega(n^2)$, but there does not exist ϵ such that $n^2 \lg n = \Omega(n^{2-\epsilon})$