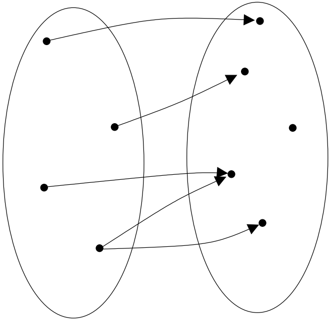
Chapter 7 outline:

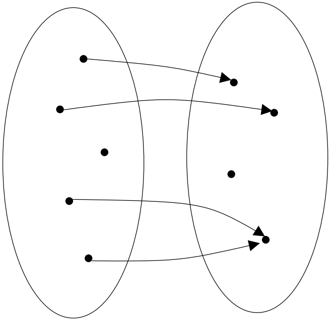
- Introduction, function equality, and anonymous functions (last week Friday)
- Image and inverse images (Monday)
- Function properties, composition, and applications to programming (**Today**)
- Cardinality (Friday)
- Countability (next week Monday)
- Review (Monday after Thanksgiving, Nov 27)
- Test 3, on Ch 6 & 7 (Wednesday after Thanksgiving, Nov 29)

Today:

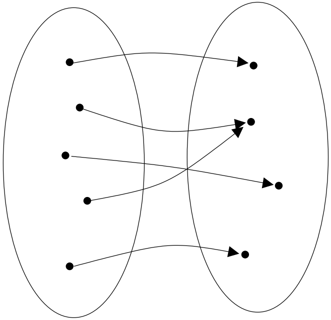
- Programming: map and filter ✓— did it last time
- Definition of one-to-one and onto, plus proofs
- Inverse functions
- Definition of function composition, plus proofs



Not a function.
(There's a domain element that is related to two things.)



Not a function.
(There's a domain element that is not related to anything.)



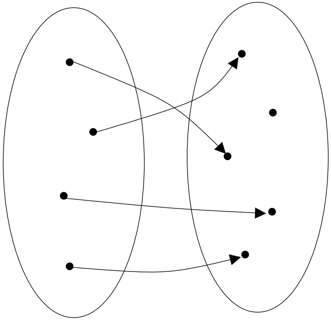
Onto (Surjection)

Everything in the codomain is hit.

$f : X \to Y$ is onto if $\forall\ y \in Y$,
$\quad \exists\ x \in X \mid f(x) = y$.

Analytic use:
f is onto.
$y \in Y$.
Hence $\exists\ x \in X$ such that $f(x) = y$.

Synthetic use:
Suppose $y \in Y$.
$\vdots$
(Somehow find x such that $f(x) = y$.)
Therefore f is onto.



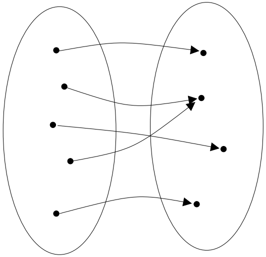
One-to-one (Injection)

Domain elements don't share.

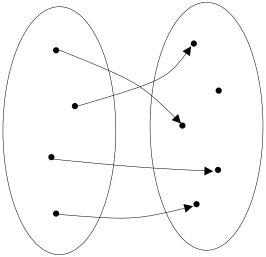
f is one-to-one if $\forall\ x_1, x_2 \in X$,
if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Analytic use:
f is one-to-one.
$f(x_1) = f(x_2)$.
Hence $x_1 = x_2$.

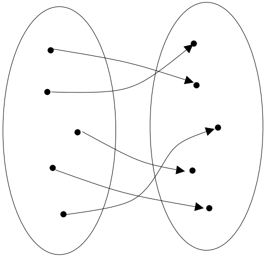
Synthetic use:
Suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$.
$\vdots$
(Somehow show $x_1 = x_2$.)
Therefore f is one-to-one.



Onto
(not one-to-one)
$|X| \geq |Y|$



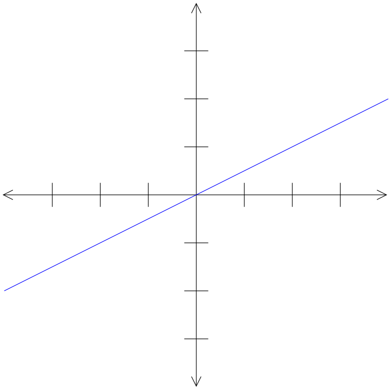
One-to-one
(not onto)
$|X| \leq |Y|$



Both onto and one-to-one

$|X| = |Y|$

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?



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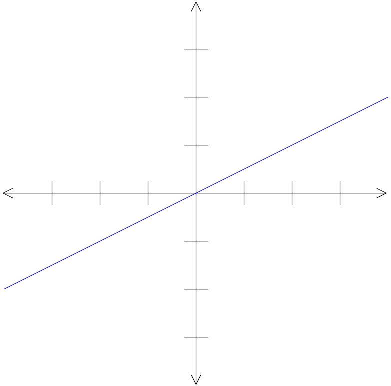

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

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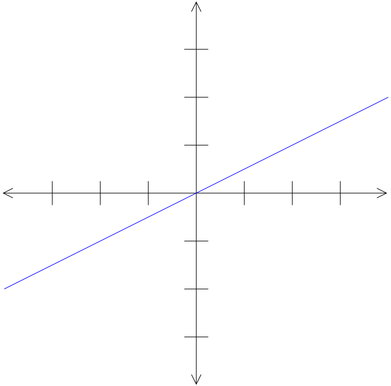

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$$\begin{aligned} \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

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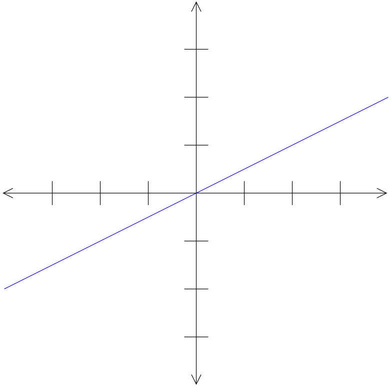

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$$\begin{aligned} \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Therefore f is one-to-one by definition. $\square$

f is onto.

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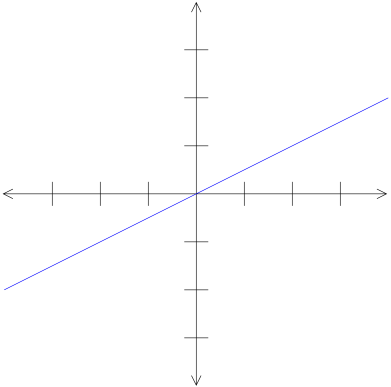

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

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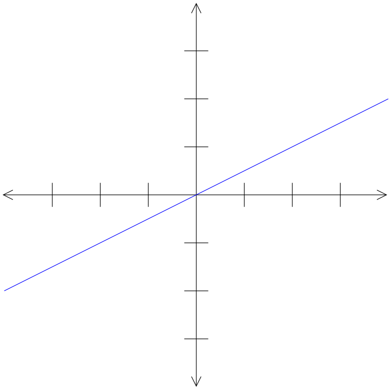
f is onto.

Proof. Suppose $y \in \mathbb{R}$. *[Want x such that $f(x) = y$.]*

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?



f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. *[Want $x_1 = x_2$]* Then, by how f is defined,

$$\begin{aligned} \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

Therefore f is one-to-one by definition. $\square$

f is onto.

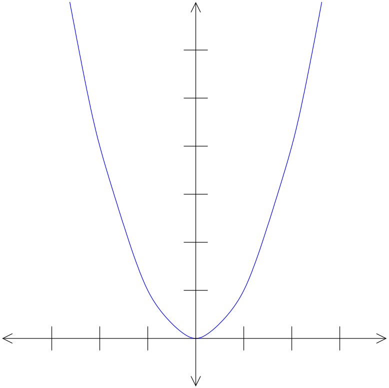
Proof. Suppose $y \in \mathbb{R}$. *[Want x such that $f(x) = y$.]*

Let $x = 2y$. Then

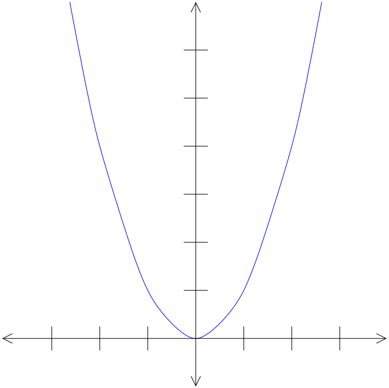
$$\begin{aligned} f(x) &= \frac{2y}{2} \\ &= y \end{aligned}$$

Therefore f is onto by definition $\square$

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Is f one-to-one? Is it onto?



Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Is f one-to-one? Is it onto?



f is not one-to-one.
$f(2) = 2^2 = 4$
$f(-2) = (-2)^2 = 4$

f is no onto.
Let $y = -1$.
$\nexists\, x \in \mathbb{R}$ such that $f(x) = -1$.

Ex 7.6.4. If $A \subseteq X$ and f is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

(Ex 7.4.9 was, Prove $A \subseteq F^{-1}(F(A))$, and Ex 7.4.10 was, Find a counterexample for $A = F^{-1}(F(A))$.)

Ex 7.6.4. If $A \subseteq X$ and f is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

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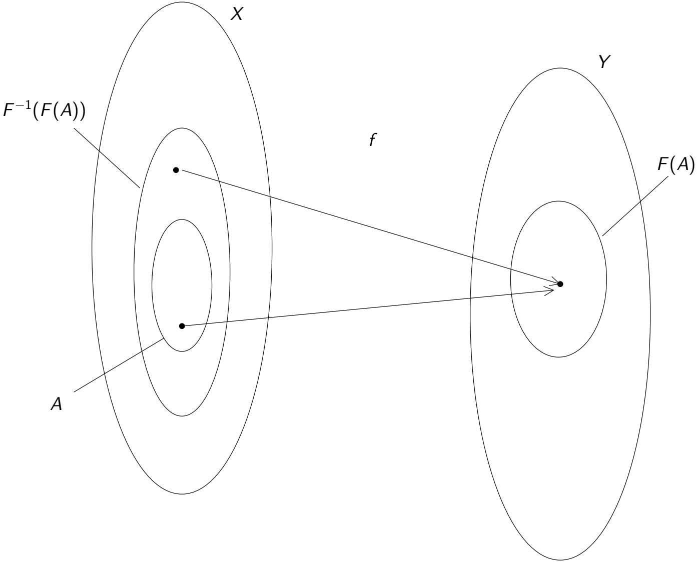
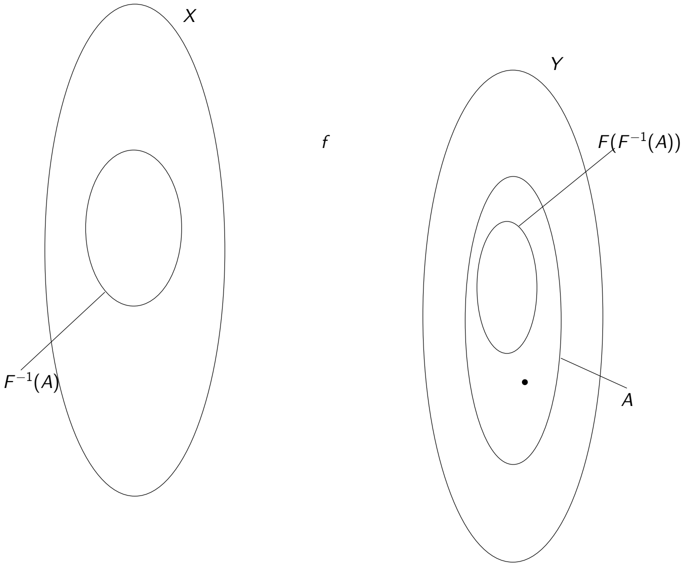


Ex 7.6.5. If $A \subseteq Y$ and f is onto, then $A \subseteq F(F^{-1}(A))$.



Inverse relation: $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$

Since a function is a relation, a function has an inverse, but we don't know that the inverse of a function **is a function**.

If $f : X \to Y$ is a **one-to-one correspondence,** then

$$f^{-1} : Y \to X = \{(y, x) \in Y \times X \mid f(x) = y\}$$

is the *inverse function* of f.

Theorem 7.8 *If $f : X \to Y$ is a one-to-one correspondence, then $f^{-1} : Y \to X$ is well defined.*

Proof. Suppose $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Hence $(y, x) \in f^{-1}$ or $f^{-1}(y) = x$.

Further suppose $(y, x_1), (y, x_2) \in f^{-1}$ (*That is, suppose that both $f^{-1}(y) = x_1$ and $f^{-1}(y) = x_2$.*) Then $f(x_1) = y$ and $f(x_2) = y$. Since f is one-to-one, $x_1 = x_2$.

Therefore, by definition of function, f^{-1} is well defined. $\square$

Relation composition: If R is a relation from X to Y and S is a relation from Y to Z, then $S \circ R$ is the relation from X to Z defined as

$$S \circ R = \{(x, z) \in X \times Z \mid \exists\, y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

$$g \circ f = \{(x, z) \in X \times Z \mid z = g(f(x))\}$$

Theorem 7.9 *If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f : X \to Z$ is well defined.*

Proof. Suppose $x \in X$. Since f is a function, there exists a $y \in Y$ such that $f(x) = y$. Since g is a function, there exists a $z \in Z$ such that $g(y) = z$. By definition of composition, $(x, z) \in g \circ f$, or $g \circ f(x) = z$.

Next suppose $(x, z_1), (x, z_2) \in g \circ f$, or $g \circ f(x) = z_1$ and $g \circ f(x) = z_2$. By definition of composition, there exist y_1, y_2 such that $f(x) = y_1$, $f(x) = y_2$, $g(y_1) = z_1$, and $g(y_2) = z_2$. Since f is a function, $y_1 = y_2$. Since g is a function, $z_1 = z_2$.

Therefore, by definition of function, $g \circ f$ is well defined. $\square$

Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

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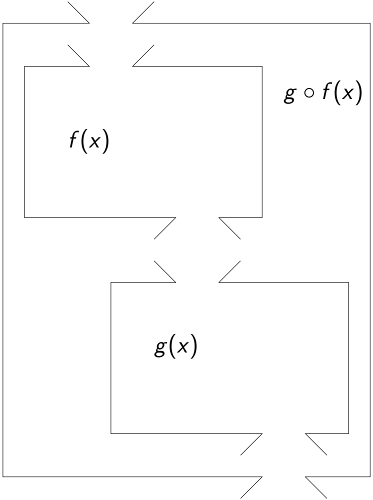


Let $f(x) = 3x$

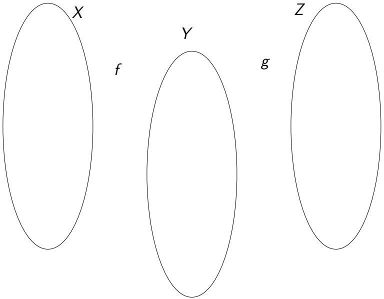
Let $g(x) = x + 7$

Then

$$\begin{aligned} g \circ f(x) &= f(x) + 7 \\ &= 3x + 7 \end{aligned}$$

Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.



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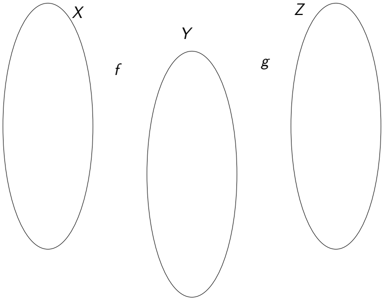
[Now, we want to prove "ontoness." Of which function?



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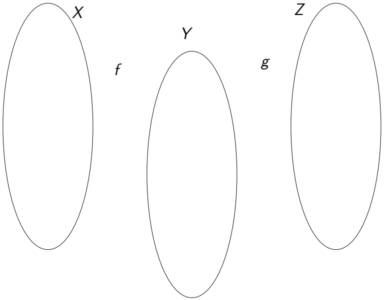

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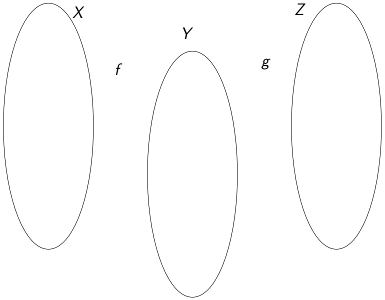
[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we're proving to be onto and show that it is hit. What is the codomain of $g \circ f$?



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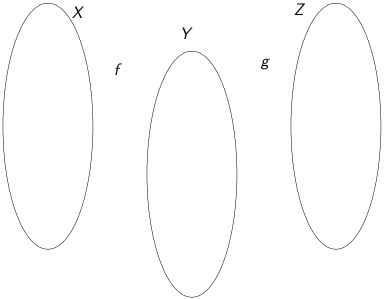
Further suppose $z \in Z$. *[We need to come up with something in the domain of $g \circ f$ that hits z. The domain is X. We will use the fact that f and g are both onto.]*



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Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove "ontoness." Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we're proving to be onto and show that it is hit. What is the codomain of $g \circ f$? Z.]

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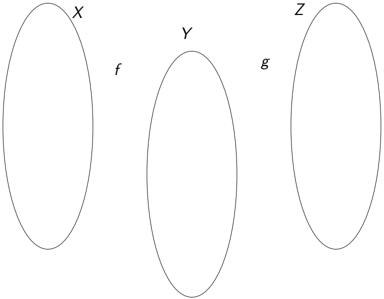
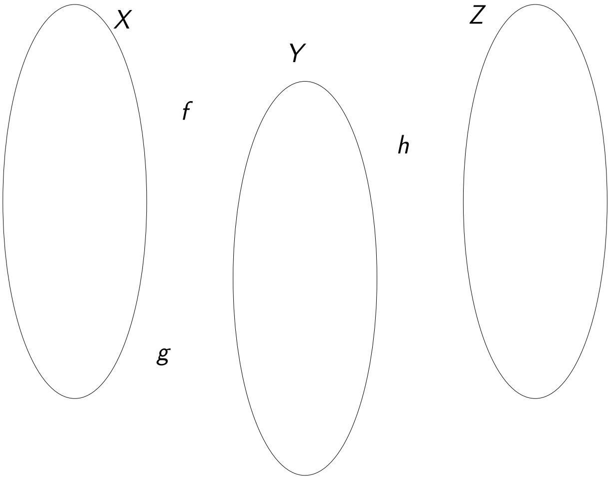


By definition of onto, there exists $y \in Y$ such that $g(y) = z$. Similarly there exists $x \in X$ such that $f(x) = y$. Now,

$$
\begin{array}{rcll}
g \circ f(x) & = & g(f(x)) & \text{by definition of function compos} \\
 & = & g(y) & \text{by substitution} \\
 & = & z & \text{by substitution}
\end{array}
$$

Therefore $g \circ f$ is onto by definition. $\square$

Ex 7.8.5. If $f : X \to Y$, $g : X \to Y$ and $h : Y \to Z$, h is one-to-one, and $h \circ f = h \circ g$, then $f = g$.



For next time:

Pg 346: 7.6.(2, 3, 6)
Ex “7.5.(a-c)” on Schoology
Pg 351: 7.8.(1, 5, 6)

Skim 7.9
Take **last** *quiz*