

Chapter 5 roadmap:

- ▶ Introduction to relations (Monday)
- ▶ Properties of relations (**Today** and Friday)
- ▶ Transitive closure (Friday, Oct 20)
- ▶ Partial order relations (Monday, Oct 23)
- ▶ Review for Test 2 (Wednesday, Oct 25)

Today and next time:

- ▶ Review of definitions from last time
- ▶ Hits on homework problems
- ▶ Properties of relations
 - ▶ Reflexivity
 - ▶ Symmetry
 - ▶ Transitivity
- ▶ Proofs
- ▶ More proofs

For next time (Fri, Oct 20):

Pg 205: 5.3.(5, 11, 14)

Pg 208: 5.4.(3, 4, 5, 22, 24, 25)

Pg 212: 5.5.(7, 9, 10)

But also (for Fri, Oct 20)...:

Read Sec 5.(6 & 7)

Take quiz

A relation from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A relation on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The image of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The image of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The inverse of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The composition of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The identity relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	=

Ex 5.3.7. Prove that if R is a relation on a set A and $(a, b) \in R$, then $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$.

Proof. Suppose R is a relation on A and that $(a, b) \in R$.

[Note that $(a, b) \in R$ implies that both a and b must be elements of A .]

Suppose $x \in \mathcal{I}_R(b)$. By definition of image, $(b, x) \in R$. Since $(a, b) \in R$, we have $(a, x) \in R \circ R$ by definition of composition. Moreover $x \in \mathcal{I}_{R \circ R}(a)$ by definition of image.

Therefore $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$ by definition of subset. \square

Ex 5.3.9. Prove that if R is a relation from A to B , then $i_B \circ R = R$.

Proof. First suppose $(x, y) \in i_B \circ R$. By definition of composition, there exists $b \in B$ such that $(x, b) \in R$ and $(b, y) \in i_B$.

By definition of the identity relation, $b = y$. By substitution, $(x, y) \in R$. Hence $i_B \circ R \subseteq R$ by definition of subset.

Next suppose $(x, y) \in R$. By how R is defined, we know $x \in A$ and $y \in B$.

By definition of the identity relation, $(y, y) \in i_B$. By definition of composition, $(x, y) \in i_B \circ R$. Hence $R \subseteq i_B \circ R$.

Therefore, by definition of set equality, $i_B \circ R = R$. \square

HW. Ex 5.3.8. Is $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$? Is $A \subseteq \mathcal{I}_{R^{-1}}(\mathcal{I}_R(A))$?

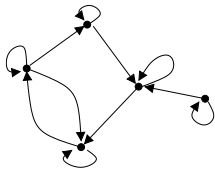
HW. Ex 5.3.10. $(R^{-1})^{-1} = R.$

Reflexivity

Informal Everything is related to itself

Formal $\forall x \in X, (x, x) \in R$

Visual

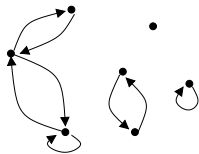


Examples $\subseteq, \leq, \geq, \equiv, i, \text{isAacquaintedWith}, \text{waterVerticallyAligned}$

Symmetry

All pairs are mutual

Formal $\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R$
OR
 $\forall (x, y) \in R, (y, x) \in R$

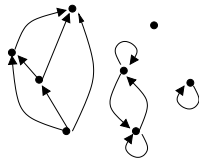


Examples $\equiv, \text{isOppositeOf}, \text{isOnSameRiver}, \text{isAacquaintedWith}$

Transitivity

Anything reachable by two hops is reachable by one hop

Formal $\forall x, y, z \in X, (x, y), (y, z) \in R \rightarrow (x, z) \in R$
OR
 $\forall (x, y), (y, z) \in R, (x, z) \in R$



Examples $<, \leq, >, \geq, \subseteq, \text{isTallerThan}, \text{isAncestorOf}, \text{isWestOf}$

Reflexivity

Formal $\forall x \in X, (x, x) \in R$

Analytical use Suppose R is reflexive
and $a \in X$.

Then $(a, a) \in R$.

Synthetic use Suppose $a \in X$.

...

$(a, a) \in R$.

Hence R is reflexive.

Symmetry

Formal $\forall x, y \in X,$
 $(x, y) \in R \rightarrow (y, x) \in R$
OR
 $\forall (x, y) \in R, (y, x) \in R$

Analytical use Suppose R is symmetric
[$a, b \in X$]
and $(a, b) \in R$.
Then $(b, a) \in R$

Synthetic use Suppose $(a, b) \in R$.

...

$(b, a) \in R$.

Hence R is symmetric.

Transitivity

Formal $\forall x, y, z \in X,$
 $(x, y), (y, z) \in R \rightarrow (x, z) \in R$
OR
 $\forall (x, y), (y, z) \in R, (x, z) \in R$

Analytical use Suppose R is transitive
[$a, b, c \in X$]
and $(a, b), (b, c) \in R$.
Then $(a, c) \in R$.

Synthetic use Suppose $(a, b), (b, c) \in R$.

...

$(a, c) \in R$.

Hence R is transitive.

Theorem 5.5. | (divides) is reflexive.

Exercise 5.4.2. \mid (divides) is not symmetric.

Theorem 5.6. $R \cap R^{-1}$ is symmetric.

Theorem 5.7. $|$ is transitive.

Exercise 5.4.20. $R^{-1} \circ R$ is reflexive. (*False*)

Exercise 5.4.21. If R and S are both reflexive, then $R \cap S$ is reflexive.

Exercise 5.4.23. If R and S are both symmetric, then $(S \circ R) \cup (R \circ S)$ is symmetric.

Based on Exercise 5.5.5. If R is transitive, then $R \circ R \subseteq R$.

Exercise 5.4.27. If R is transitive, $\mathcal{I}_R(\mathcal{I}_R(A)) \subseteq \mathcal{I}_R(A)$.

Exercise 5.5.4. If R is reflexive and

(for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(c, a) \in R$),
then R is an equivalence relation.

Exercise 5.5.8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation. (*True or false?*)

Exercise 5.5.6. If R is an equivalence relation and $(a, b) \in R$, then $\mathcal{I}_R(a) = \mathcal{I}_R(b)$.

For next time (Fri, Oct 20):

Pg 205: 5.3.(5, 11, 14)

Pg 208: 5.4.(3, 4, 5, 22, 24, 25)

Pg 212: 5.5.(7, 9, 10)

But also (for Fri, Oct 20)...:

Read Sec 5.(6 & 7)

Take quiz