

Chapter 5 roadmap:

- ▶ Introduction to relations (Monday before break)
- ▶ Properties of relations (Wednesday and Friday before break)
- ▶ Transitive closure (**Today**)
- ▶ Partial order relations (next week Monday)
- ▶ Review for Test 2 (next week Wednesday)
- ▶ Test 2 on Chapters 4 & 5 (next week Friday)

Today:

- ▶ Review of relation properties
- ▶ An arithmetic on relations
- ▶ Computing whether a function is transitive
- ▶ Transitive closure

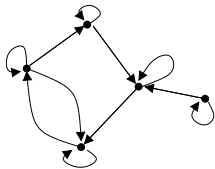
A relation from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A relation on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The image of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The image of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The inverse of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The composition of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The identity relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	=

Reflexivity

Informal Everything is related to itself

Formal $\forall x \in X, (x, x) \in R$

Visual

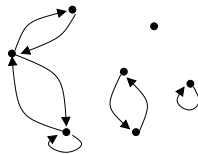


Examples $\subseteq, \leq, \geq, \equiv, i, \text{isAacquaintedWith}, \text{waterVerticallyAligned}$

Symmetry

All pairs are mutual

Formal $\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R$
OR
 $\forall (x, y) \in R, (y, x) \in R$

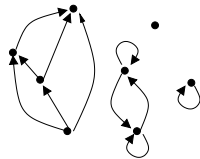


Examples $\equiv, \text{isOppositeOf}, \text{isOnSameRiver}, \text{isAacquaintedWith}$

Transitivity

Anything reachable by two hops is reachable by one hop

Formal $\forall x, y, z \in X, (x, y), (y, z) \in R \rightarrow (x, z) \in R$
OR
 $\forall (x, y), (y, z) \in R, (x, z) \in R$



Examples $<, \leq, >, \geq, \subseteq, \text{isTallerThan}, \text{isAncestorOf}, \text{isWestOf}$

Operators

$$x + y$$
$$\neg x$$

$$p \vee q$$
$$\sim p$$

$$A \cup B$$
$$\overline{A}$$

Distribution

$$x \cdot (y + z)$$
$$= x \cdot y + x \cdot z$$

$$p \wedge (q \vee r)$$
$$\equiv (p \wedge q) \vee (p \wedge r)$$

$$A \cap (B \cup C)$$
$$= (A \cap B) \cup (A \cap C)$$

Identity

$$x + 0 = x$$
$$x \cdot 1 = x$$

$$p \vee T \equiv p$$
$$p \wedge F \equiv p$$

$$A \cup \emptyset = A$$
$$A \cap \mathcal{U} = A$$

$$S \circ R$$

$$R^{-1}$$

$$i_X \circ R = R$$

$$R^2 = R \circ R$$

R is one less than

eats

is parent of

R^2 is two less than

eats something that
eats

is grandparent of

R^3 is three less than

eats something that
eats something that
eats

is great grandparent of

???

<

gets nutrients from

is ancestor of

Definition of transitivity

Short form: $\forall (x, y), (y, z) \in R, (x, z) \in R$

Transform this to:

$$\forall (x, y) \in R, \forall (w, z) \in R, \text{ if } y = w \text{ then } (x, z) \in R$$

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Definition of transitivity

Short form: $\forall (x, y), (y, z) \in R, (x, z) \in R$

Transform this to:

$$\forall (x, y) \in R, \forall (w, z) \in R, \text{ if } y = w \text{ then } (x, z) \in R$$

$\{(1, 2), (2, 3), (5, 2), (1, 5), (2, 5), (1, 3)\}$

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Computing transitivity is a $\forall \forall \exists$ problem

Our strategy is, for each pair (x, y) , walk through the whole (original) list. If the list

1. is empty, then true (vacuously)
2. begins with (y, z) (that is, begins with (w, z) where $y = w$), then search the whole (original) list for (x, z) .
 - 2.1 if found, keep searching
 - 2.2 if not found, then false
3. begins with (w, z) for $w \neq y$, skip it and keep searching

Domain

Rivers

First relation*flows into*

The Platte flows into the Missouri, and the Missouri flows into the Mississippi.

Second relation*is tributary to*

The Platte is a tributary to the Missouri; both the Platte and the Missouri are tributaries to the Mississippi.

People

is parent of

Bill is Jane's parent; Jane is Leroy's parent

is ancestor of

Bill is Jane's ancestor; Leroy has both Jane and Bill as ancestors.

Domain

Animals

First relation*eats*

Rabbit eats clover; coyote eats rabbit.

Second relation*derives nutrients from*

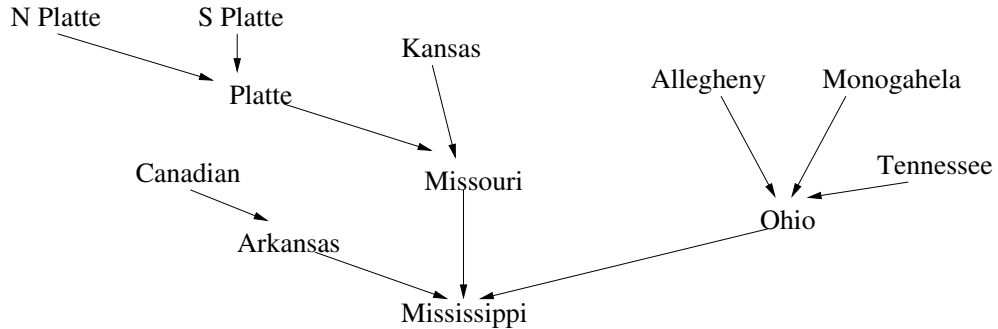
Coyote derives nutrients from rabbit; rabbit derives nutrients from clover; both coyote and rabbit ultimately derive nutrients from clover.

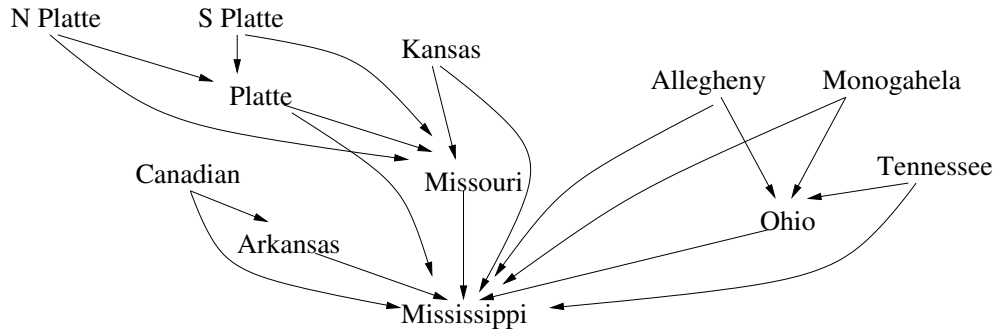
 \mathbb{Z} *is one less than*

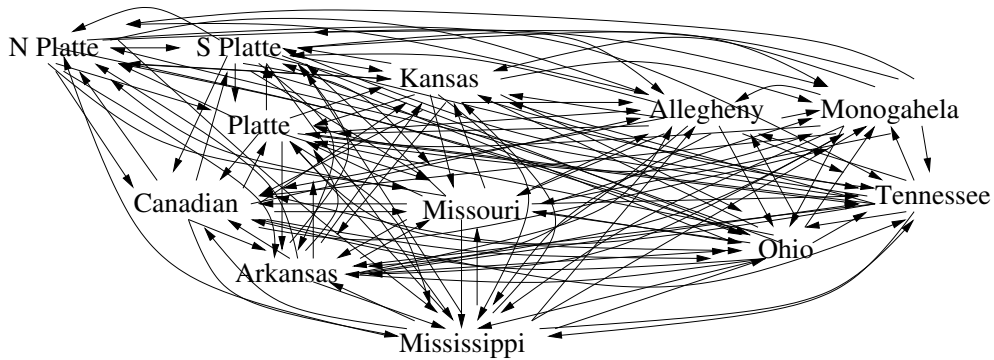
2 is one less than 3; 3 is one less than 4

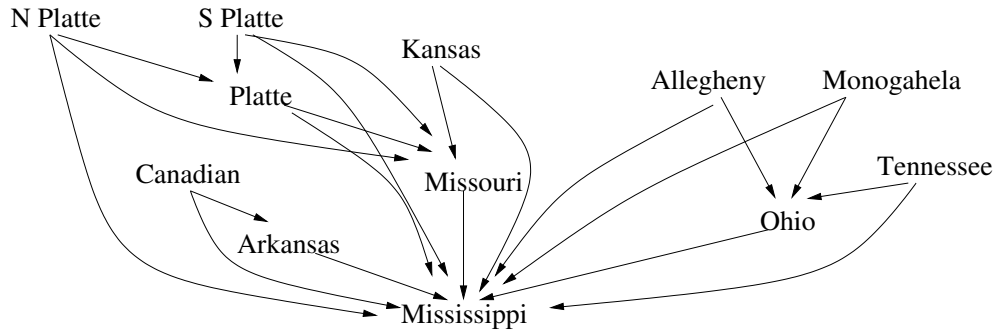
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$2 < 3$; $3 < 4$; $2 < 4$.









If R is a relation on X , then R^T is the **transitive closure** of R if

- ▶ R^T is transitive
- ▶ $R \subseteq R^T$
- ▶ If S is a transitive relation such that $R \subseteq S$, then $R^T \subseteq S$

Which of the following expresses a transitive closure?

- ▶ My friends are my friends, and no one else.
- ▶ Any friend of my friend is also my friend.
- ▶ Any friend of my friends' friends is also my friend.
- ▶ My friends are my friends, and so are my friends's friends, and so are my friends' friends' friends, and so on forever.

Let R be a relation and let T be the transitive closure of R . What, then, do you know to be true? Select all that apply.

- ▶ R is transitive
- ▶ T is a proposition
- ▶ T is a relation
- ▶ T is transitive
- ▶ T is a powerset
- ▶ $R \subseteq T$
- ▶ $T \subseteq R$

Theorem 5.12 *The transitive closure of a relation R is unique.*

Proof. *Suppose S and T are relations fulfilling the requirements for being transitive closures of R . By items 1 and 2, S is transitive and $R \subseteq S$, so by item 3, $T \subseteq S$. By items 1 and 2, T is transitive and $R \subseteq T$, so by item 3, $S \subseteq T$. Therefore $S = T$ by the definition of set equality. \square*

Other closures:

Ex 5.7.2 $R \cup i_A$ is the reflexive closure of R

Ex 5.7.3. $R \cup R^{-1}$ is the symmetric closure of R . (HW)

Ex 5.7.2 $R \cup i_A$ is the reflexive closure of R

Proof. Suppose R is a relation on A .

[$R \cup i_A$ is reflexive:] Suppose $a \in A$. $(a, a) \in i_A$ by definition of identity relation. $(a, a) \in R \cup i_A$ by definition of union. Hence $R \cup i_A$ is reflexive by definition.

[$R \subseteq R \cup i_A$:] Suppose $(a, b) \in R$. Then $(a, b) \in R \cup i_A$ by definition of union. Hence $R \subseteq R \cup i_A$. (Alternately, we could have cited Exercise 4.2.1.)

[$R \cup i_A$ is the smallest such relation:] Suppose S is a reflexive relation such that $R \subseteq S$. Suppose further $(a, b) \in R \cup i_A$. By definition of union, $(a, b) \in R$ or $(a, b) \in i_A$.

Case 1: Suppose $(a, b) \in R$. Then $(a, b) \in S$ by definition of subset (since we supposed $R \subseteq S$).

Case 2: Suppose $(a, b) \in i_A$. Then, by definition of identity relation, $a = b$. $(a, a) \in S$ by definition of reflexive (since we suppose S is reflexive). $(a, b) \in S$ by substitution.

Either way, $(a, b) \in S$ and hence $R \cup i_A \subseteq S$ by definition of subset.

Therefore, $R \cup i_A$ is the reflexive closure of R . \square

Theorem 5.13 *If R is a relation on a set A , then*

$$R^\infty = \bigcup_{i=1}^{\infty} R^i = \{(x, y) \mid \exists i \in \mathbb{N} \text{ such that } (x, y) \in R^i\}$$

is the transitive closure of R .

Proof. *Suppose R is a relation on a set A .*

Suppose $a, b, c \in A$, $(a, b), (b, c) \in R^\infty$. By the definition of R^∞ , there exist $i, j \in \mathbb{N}$ such that $(a, b) \in R^i$ and $(b, c) \in R^j$. By the definition of relation composition and Exercise 5.7.4, $(a, c) \in R^j \circ R^i = R^{i+j}$. $R^{i+j} \subseteq R^\infty$ by the definition of R^∞ . By the definition of subset, $(a, c) \in R^\infty$. Hence, R^∞ is transitive by definition.

Suppose $a, b \in A$ and $(a, b) \in R$. By the definition of R^∞ (taking $i = 1$), $(a, b) \in R^\infty$, and so $R \subseteq R^\infty$, by definition of subset.

Suppose S is a transitive relation on A and $R \subseteq S$. Further suppose $(a, b) \in R^\infty$. Then, by definition of R^∞ , there exists $i \in \mathbb{N}$ such that $(a, b) \in R^i$. By Lemma 5.14, $(a, b) \in S$. Hence $R^\infty \subseteq S$ by definition of subset.

Therefore, R^∞ is the transitive closure of R . \square

For next time:

Pg 217: 5.6.(1 & 3)

Pg 222: 5.7.(3,4,5)

For Exercise 5.7.4, it should say $(S \circ R) \circ Q = S \circ (R \circ Q)$ instead of $(R \circ S) \circ Q = R \circ (S \circ Q)$.

Read 5.(8 & 9)

Take quiz