

Chapter 6 outline:

- ▶ Introduction, function equality, and anonymous functions (week-before Wednesday)
- ▶ Image and inverse images (week-before Friday)
- ▶ Function properties and composition (last week Monday)
- ▶ Map, reduce, filter (last week Wednesday)
- ▶ Cardinality (last week Friday)
- ▶ Countability **plus practice quiz (Today)**
- ▶ Review (Wednesday)
- ▶ Test 3, on Ch 5 & 6 (Friday)

- ▶ $A \subseteq B$ iff $(B - A) \cup A = B$.
- ▶ $(B - A) \cap A = \emptyset$
- ▶ If A and B are finite, disjoint sets, then $|A \cup B| = |A| + |B|$.

Assume A and B are finite sets.

Ex 6.6.1. If $A \subseteq B$, then $|B - A| = |B| - |A|$.

Ex 6.6.2. If $A \subseteq B$, then $|A| \leq |B|$.

Two finite sets X and Y have the *same cardinality* as each other if there exists a one-to-one correspondence from X to Y .

To use this *analytically*:

Suppose X and Y have the same cardinality. Then let f be a one-to-one correspondence from X to Y .

f is both onto and one-to-one.

To use this *synthetically*:

Given sets X and $Y \dots$

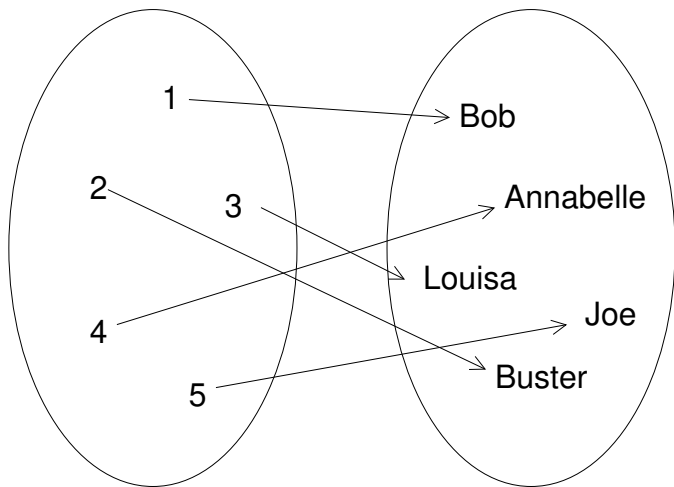
[Define f] Let $f : X \rightarrow Y$ be a function defined as \dots

Suppose $y \in Y$. *Somehow find* $x \in X$ such that $f(x) = y$. Hence f is onto.

Suppose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. *Somehow show* $x_1 = x_2$. Hence f is one-to-one.

Since f is a one-to-one correspondence, X and Y have the same cardinality as each other.

A finite set X has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \dots, n\}$ to X . Moreover, $|\emptyset| = 0$.



Two finite sets X and Y have the *the same cardinality* as each other if there exists a one-to-one correspondence from X to Y .

A finite set X has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \dots, n\}$ to X . Moreover, $|\emptyset| = 0$.

Given a set X , if there exists $n \in \mathbb{N}$ and a one-to-one correspondence from $\{1, 2, \dots, n\}$ to X , then X is *finite* and $|X| = n$. Otherwise, X is *infinite*.

Are all infinities equal?

Which is more intuitive to you,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{R}|$$

or

$$|\mathbb{N}| < |\mathbb{W}| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|$$

?

Thm 6.15. \mathbb{W} and \mathbb{N} have the same cardinality.

Proof. [We need a one-to-one correspondence from \mathbb{N} to \mathbb{W} .]

Let $f : \mathbb{W} \rightarrow \mathbb{N}$ be defined so that $f(n) = n + 1$.

Suppose $n \in \mathbb{N}$. Then $f(n - 1) = (n - 1) + 1 = n$, so f is onto.

Next suppose $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$, and moreover $n_1 = n_2$. Hence f is one-to-one.

Since a one-to-one correspondence exists between \mathbb{W} and \mathbb{N} , the two sets have the same cardinality. \square

A set X is *countably infinite* if there exists a one-to-one correspondence from \mathbb{N} to X .

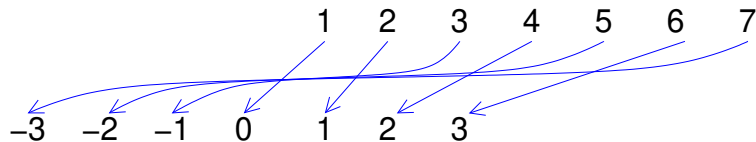
A set is *countable* if it is finite or countably infinite. Otherwise it is *uncountable*.

Thm 6.16. \mathbb{Z} is countably infinite.

Proof. [We need a one-to-one correspondence from \mathbb{N} to \mathbb{Z} .]

Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be defined so that

$$f(x) = \begin{cases} n \operatorname{div} 2 & \text{if } n \text{ is even} \\ -(n \operatorname{div} 2) & \text{otherwise} \end{cases}$$



Since f is a one-to-one correspondence, \mathbb{Z} is countably infinite. \square

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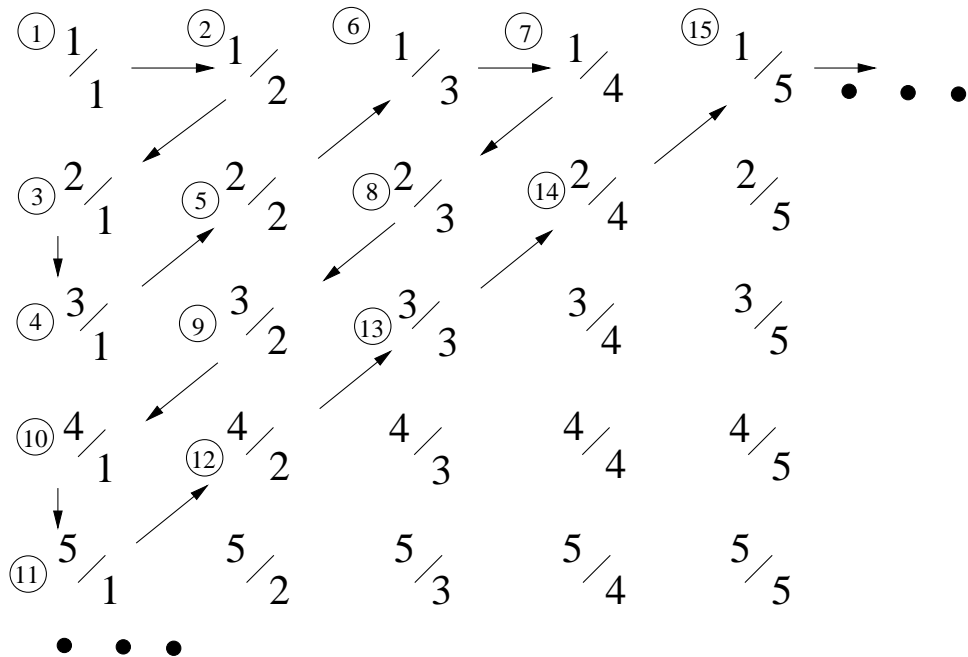
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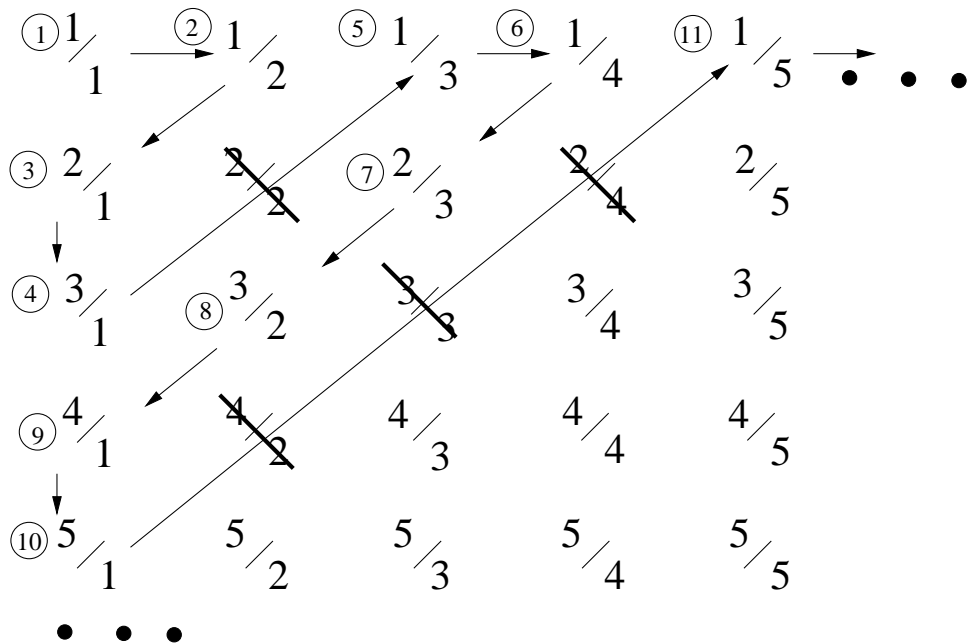
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def next_seat(bus_number, seat_number) :
    return ((1, seat_number+1) if bus_number == 1 and seat_number%2==1
            else (bus_number+1, 1) if seat_number == 1 and bus_number%2==0
            else (bus_number-1, seat_number+1) if (seat_number + bus_number)%2==0
            else (bus_number+1, seat_number-1))

def bus_seat_to_hotel_room(bus_number, seat_number) :
    return len(generate_until([(1,1)], lambda ss : next_seat(* (ss[-1])),
                               lambda ss: ss[-1] == (bus_number, seat_number)))

def hotel_room_to_bus_seat(hotel_room) :
    return generate_until([(1,1)], lambda ss : next_seat(* (ss[-1])),
                           lambda ss: len(ss) == hotel_room)[-1]
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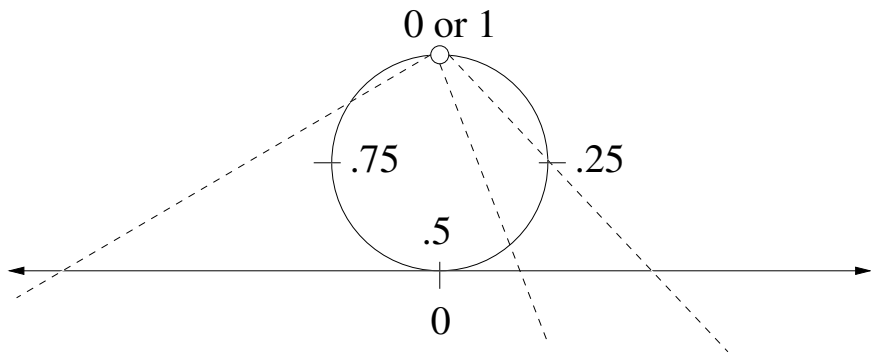
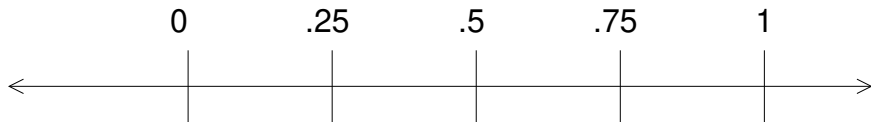
Thm 6.17. \mathbb{Q}^+ is countably infinite.

So,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}|$$

What about \mathbb{R} ?

Thm 6.18. $(0, 1)$ has the same cardinality as \mathbb{R} .



Thm 6.19. $(0, 1)$ is uncountable.

Proof. Suppose $(0, 1)$ is countable. Then there exists a one-to-one correspondence $f : \mathbb{N} \rightarrow (0, 1)$. We will use f to give names to all the digits of all the numbers in $(0, 1)$, considering each number in its decimal expansion, where each $a_{i,j}$ stands for a digit.:

$$\begin{aligned} f(1) &= 0.a_{1,1}a_{1,2}a_{1,3} \dots a_{1,j} \dots \\ f(2) &= 0.a_{2,1}a_{2,2}a_{2,3} \dots a_{2,j} \dots \\ &\vdots \\ f(x) &= 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,j} \dots \\ &\vdots \end{aligned}$$

Now construct a number $d = 0.d_1d_2d_3 \dots d_i \dots$ as follows

$$d_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1 \\ 2 & \text{if } a_{i,i} = 1 \end{cases}$$

Since $d \in (0, 1)$ and f is onto, there exists an $x \in \mathbb{N}$ such that $f(x) = d$.
Moreover,

$$f(x) = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

so

$$d = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

by substitution. In other words, $d_i = a_{x,i}$, and specifically $d_x = a_{x,x}$. However, by the way that we have defined d , we know that $d_x \neq a_{x,x}$, a contradiction. Therefore $(0, 1)$ is not countable. \square

For next time:

(No HW exercises)

Take quiz on cardinality and countability