Chapter 7 outline:

- Recursively-defined sets (last week Monday)
- Structural induction (**Today**)
- Mathematical induction (Wednesday)
- Non-recursive programs—loops (Friday)
- Loop invariant proofs (next week Monday)
- A language processor (next week Wednesday)

Last time we saw

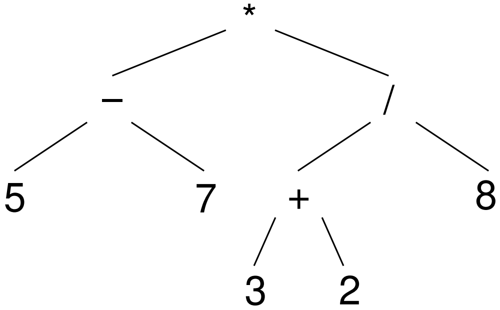
- A recursive definition of whole numbers
- A recursive definition of trees, particularly *full binary trees*; a full binary tree is either
  - a leaf, or
  - an internal node together with two children which are full binary trees.

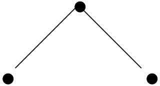
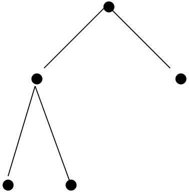
Today we see

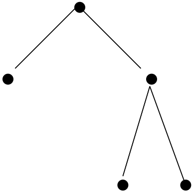
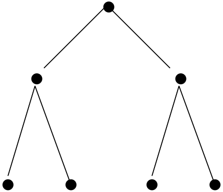
- Self-referential proofs

Expression trees:

$$
\begin{aligned}
\textit{Expression} &\rightarrow \textit{Variable} \mid \textit{Constant} \\
&\quad \mid \textit{Expression}\,\textit{Operator}\,\textit{Expression} \\
\\
\textit{Operator} &\rightarrow + \mid - \mid * \mid /
\end{aligned}
$$



Tree	Nodes	Links
	1	0
	3	2
	5	4

Tree	Nodes	Links
	5	4
	7	6

While building bigger trees from smaller trees, *the number of nodes is (and remains) one more than the number of links*. (Invariant)

Theorem 7.1 *For any full binary tree T,* $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Let $\mathcal{T}$ be the set of full binary trees. Then, we're saying

$$\forall\ T \in \mathcal{T}, \texttt{nodes}(T) = \texttt{links}(T) + 1$$

Theorem 7.1 *For any full binary tree T, $\texttt{nodes}(T) = \texttt{links}(T) + 1$.*

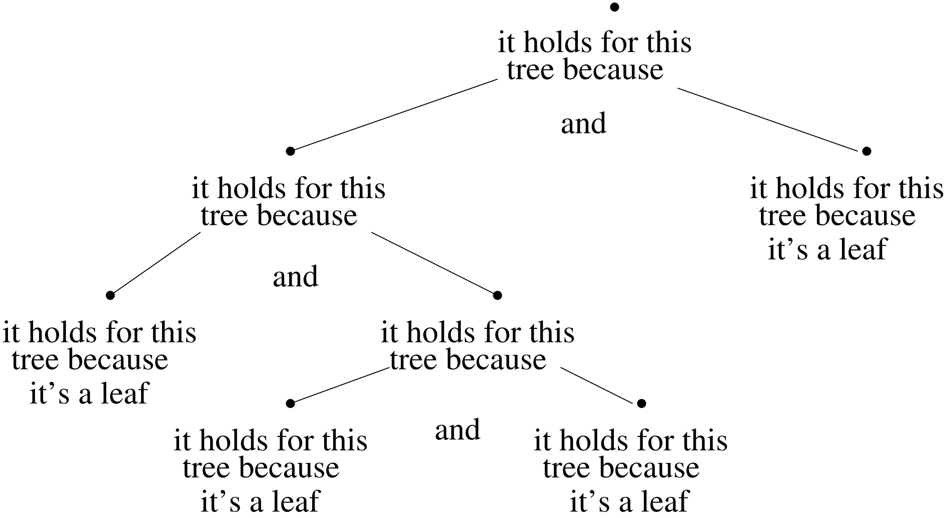
Proof. Suppose $T \in \mathcal{T}$. *[What is a tree? the definition says it's either a leaf or an internal with two subtrees. We can use division into cases.]*

Case 1. Suppose T is a leaf. Then, by how `nodes` and `links` are defined, $\texttt{nodes}(T) = 1$ and $\texttt{links}(T) = 0$. Hence $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Case 2. Suppose T is an internal node with links to subtrees T_1 and T_2. Moreover, by how `nodes` and `links` are defined, $\texttt{links}(T) = \texttt{links}(T_1) + \texttt{links}(T_2) + 2$. Then,

$$\begin{aligned}
\texttt{nodes}(T) &= 1 + \texttt{nodes}(T_1) + \texttt{nodes}(T_2) && \text{by the definition of } \texttt{nodes} \\
&= 1 + \texttt{links}(T_1) + 1 + \texttt{links}(T_2) + 1 && \text{by Theorem 7.1} \\
&= \texttt{links}(T_1) + \texttt{links}(T_2) + 2 + 1 && \text{by algebra} \\
&= \texttt{links}(T) + 1 && \text{by substitution}
\end{aligned}$$

Either way, $\texttt{nodes}(T) = \texttt{links}(T) + 1$. □


it holds for this tree because
and
it holds for this tree because
and
it holds for this tree because it's a leaf
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it holds for this tree because it's a leaf
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it holds for this tree because it's a leaf
it holds for this tree because it's a leaf

Theorem 7.1 *For any full binary tree T, $\texttt{nodes}(T) = \texttt{links}(T) + 1$.*

Proof. Suppose $T \in \mathcal{T}$.

Base case. Suppose T is a leaf. Then, by how nodes and links are defined, $\texttt{nodes}(T) = 1$ and $\texttt{links}(T) = 0$. Hence $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Inductive case Suppose T is an internal node with links to subtrees T_1 and T_2 such that $\texttt{nodes}(T_1) = \texttt{links}(T_1) + 1$ and $\texttt{nodes}(T_2) = \texttt{links}(T_2) + 1$. Moreover, by how nodes and links are defined, $\texttt{links}(T) = \texttt{links}(T_1) + \texttt{links}(T_2) + 2$. Then,

$$\begin{array}{rcll}
\texttt{nodes}(T) & = & 1 + \texttt{nodes}(T_1) + \texttt{nodes}(T_2) & \text{by the definition of nodes} \\
& = & 1 + \texttt{links}(T_1) + 1 + \texttt{links}(T_2) + 1 & \text{by the inductive hypothesis} \\
& = & \texttt{links}(T_1) + \texttt{links}(T_2) + 2 + 1 & \text{by algebra} \\
& = & \texttt{links}(T) + 1 & \text{by substitution}
\end{array}$$

Either way, $\texttt{nodes}(T) = \texttt{links}(T) + 1$. □

Let X be a a recursively defined set, and let $\{Y, Z\}$ be a partition of X, where Y is defined by a simple set of elements $Y = \{y_1, y_2, \ldots\}$ and Z is defined by a recursive rule.

Examples:

- X is the set of pizzas, $Y = \mathit{Crusts}$, and $Z = \{(\mathit{top}, \mathit{bot}) \mid \mathit{top} \in \mathit{Toppings} \ \wedge \ \mathit{bot} \in X\}$
- $X = \mathbb{W}$, $Y = \{0\}$, and $Z = \{\mathtt{succ}(n) \mid n \in \mathbb{W}\}$
- $X = \mathcal{T}$, Y is the set of leaves, and Z is the set of internals with children $T_1, T_2 \in \mathcal{T}$.

Let X be a a recursively defined set, and let $\{Y, Z\}$ be a partition of X, where Y is defined by a simple set of elements $Y = \{y_1, y_2, \ldots\}$ and Z is defined by a recursive rule.

To prove something in the form of $\forall\ x \in X, I(x)$, do this:

Base case: Suppose $x \in Y$.

$\vdots$

$I(x)$

Inductive case: Suppose $x \in Z$. *[Using x and the definition of Z, find components $a, b, \ldots \in X$.]*

Suppose $I(a), I(b), \ldots$. *[The* **inductive hypothesis***]*

$\vdots$

Use the inductive hypothesis

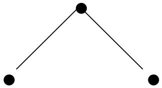
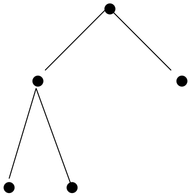
$\vdots$

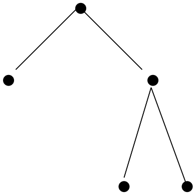
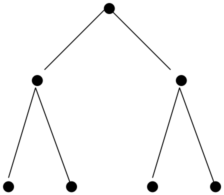
$I(x)$ $\Box$

7.2.1 For any full binary tree T, $\texttt{leaves}(T) = \texttt{internals}(T) + 1$.

Let the *height* of a full binary tree be 1 if the tree is a node by itself (leaf), or 1 more than the maximum height of its two children, if it is an internal node.

7.2.5 For any full binary tree T, $\texttt{nodes}(T) \leq 2^{\texttt{height}(T)} - 1$.

Tree	Nodes	Height
	1	1
	3	2
	5	3

Tree	Nodes	Height
	5	3
	7	3

For next time:

First, read Sections 7.(1 & 2), from today and last time.

Then do Exercises 7.1.(1-5) and 7.2.(2,3,5).

Then read Section 7.3

Take quiz