Exit strategy:

- Exact Cover: reduction from SAT (pg 318)
- Ham Cycle: reduction from Exact Cover (p 320)
- HamPath: reduction from Ham Cycle (Ex 7.3.3)
- Undirected Ham Cycle: reduction from Ham Cycle (pg 323)
- TSP: reduction from Uni Ham Cycle (pg 324)
- Knapsack: reduction from Exact Cover (pg 325)
- Indep Set: reduction from 3-SAT (pg 326)
- Clique: reduction from Indep Set (pg 327)
- Longest Cycle: reduction from Ham Cycle (7.3.4.a)
- Subgraph Isomorphism: reduction from Ham Cycle (7.3.4.b)

Definition 7.1.2.: A language $L \subseteq \Sigma*$ is $\mathcal{NP}$-**complete** if

1. $L \in \mathcal{NP}$
2. For every language $L' \in \mathcal{NP}$, there is a polynomial reduction from L' to L [L is $\mathcal{NP}$-**hard**.

Let $\mathcal{NPC}$ be the class of $\mathcal{NP}$-complete languages.

Theorem 7.1.1: $\mathcal{P} = \mathcal{NP}$ iff $\exists$ $L \in \mathcal{NPC}$ such that $L \in \mathcal{P}$.

Proving that a problem is $\mathcal{NP}$-complete shows that it is at least as hard as all the other problems shown to be $\mathcal{NP}$-complete.

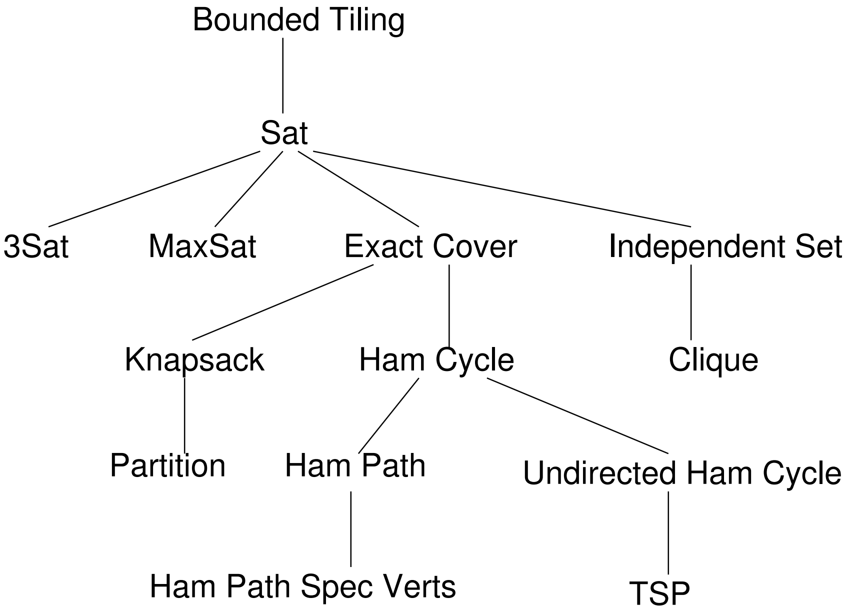
A. Prove $L \in \mathcal{NP}$

1. Describe a certificate.
2. Demonstrate it can be used to check a string/solution in polynomial time.
3. Demonstrate that the certificate iteslef is succinct (polynomial in size)

} usually easy for our problems—ok to do briefly/informally

B. Prove L is $\mathcal{NP}$-hard

1. Choose a known $\mathcal{NP}$-complete problem L_2.
2. Describe a reduction τ from L_2 to L.
3. Demonstrate τ can be computed in polynomial time. (Also usually easy.)
4. Demonstrate that $x \in L_2$ iff $\tau(x) \in L$


Bounded Tiling
Sat
3Sat
MaxSat
Exact Cover
Independent Set
Knapsack
Ham Cycle
Clique
Partition
Ham Path
Undirected Ham Cycle
Ham Path Spec Verts
TSP

Reducing Sat to Exact Cover (Given $\mathcal{U}$, set of set $\mathcal{F} \subseteq \mathscr{P}(\mathcal{U})$, find partition):
Suppose $\{c_1, c_2, \dots c_\ell\}$ is an instance of Sat.
Define the following instance of Exact Cover:

$$\mathcal{U} = \begin{array}{cll} & \{x_i\} & \text{for each variable } i \\ \cup & \{c_j\} & \text{for each clause } j \\ \cup & \{p_{jk}\} & \text{for each position } k \text{ in clause } j \end{array}$$

$$\mathcal{F} = \begin{array}{llll} \{ & \forall j,k & \{p_{jk}\} & \\ & \forall i & T_{i\top} = \{x_i\} \cup \{p_{jk} \mid \lambda_{jk} = \sim x_i\} & \\ & \forall i & T_{i\bot} = \{x_i\} \cup \{p_{jk} \mid \lambda_{jk} = x_i\} & \\ & \forall j,k & \{c_j p_{jk}\} & \} \end{array}$$

- At least one of $T_{i\bot}$ or $T_{i\top}$ for each i must be in the cover, which stands for the truth assignment.
- At least one of $\{c_j p_{jk}\}$ must be in the cover, which stands for which literal satisfies clause j.
- The extra $\{p_{jk}\}$ sets can be chosen as necessary to account for literals not used in satisfying the formula.

$S_1 = \{u_3, u_4\}$

$S_2 = \{u_2, u_3, u_4\}$

$S_3 = \{u_1, u_2\}$

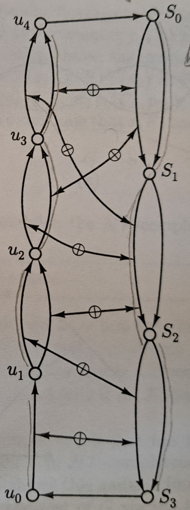


Figure 7-6

Proof that HAMILTONPATH is $\mathcal{NP}$-Complete

Proof. *[HAMILTONPATH is $\mathcal{NP}$.] Suppose $G = (V, E)$ is a graph, an instance of the HAMILTONPATH. Let $p = \langle u_1, u_2, \ldots u_n \rangle$ be a a sequence of vertices from V, a proposed Hamilton path in G. With any reasonable representation of G, one can check that each vertex in V appears uniquely in p, and that for any pair of vertices u_i, u_{i+1} as they appear in p, the edge (u_i, u_{i+1}) is in E. Moreover, the path p is smaller than the representation of G, so it is succinct.*

[HAMILTONPATH is $\mathcal{NP}$-hard.] Next, suppose $G = (E, V)$ is an instance of HAMILTONCYCLE. Let $v_1 \in V$ be an arbitrary vertex. Let $G' = (V', E')$ be a new graph such that v_1 is removed and four new vertices are added, that is, $V' = V - \{v_1\} \cup \{v_a, v_b, v_c, v_d\}$; and every edge that is incident on v_1 is replaced with two analogous edges incident on v_b and v_c, and and edges (v_a, v_b) and (v_c, v_d) are added, that is

$$
\begin{aligned}
E' \quad = \quad & (E - \{(v_1, v_x) \mid (v_1, v_x) \in E\}) \\
& \cup \{(v_b, v_x), (v_c, v_x) \mid (v_1, v_x) \in E\} \\
& \cup \{(v_a, v_b), (v_c, v_d)\}
\end{aligned}
$$

This reduction reduction is accomplished by one pass over the edges, which is polynomially computable.
Now, suppose G has a Hamilton cycle, call it $(v_1, v_2, \ldots v_{|V|-1}, v_1)$*. (As a cycle, it has an arbitrary starting/ending point, so we are free to choose* v_1 *as the starting point when naming the cycle.) Then* G' *has a Hamiltonian path* $(v_a, v_b, v_2, \ldots, v_{|V|-1}, v_c, v_d)$*.*
Conversely, suppose G' *has a Hamiltonian path. Based on how we constructed* G' *(for example, the only edge going out of* v_a *is* (v_a, v_b)*, and the only edge going into* v_d *is* (v_c, v_d)*), that path must be in the form* $(v_a, v_b, v_2, \ldots, v_{|V|-1}, v_c, v_d)$*. Then G has a Hamiltonian cycle* $(v_1, v_2, \ldots v_{|V|-1}, v_1)$*.*
Therefore HAMILTON PATH *is* $\mathcal{NP}$*-complete.* □

Reduction from UHC to TSP (LP pg 324).

Differences between UHC and TSP:

- The graph in TSP is *weighted* (interpreted as distances)
- The graph in TSP is *complete*
- A TSP problem has a *budget*

Suppose we have an instance of UHC, an undirected graph $G = (V, E)$. Construct a graph with the same vertices but complete in its edges and with distances

$$d_{i,j} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$$

Set the budget to $|V|$.

Reduction from EXACT COVER to KNAPSACK (LP pg 325).

Given an instance of EXACT COVER $(\mathcal{U}, \mathcal{F} \subseteq \mathscr{P}(\mathcal{U}))$, construct an instance of KNAPSACK (S, K):

- $S = \{bit\_vec(S_i) \mid S_i \in \mathcal{F}\}$ where *bit_vec* computes the bit-vector representation of a set.
- $K = 2^{|\mathcal{U}|} - 1 = \sum_{i=0}^{|\mathcal{U}|-1} 2^i$

Interpret each set in $\mathcal{P}(S)$ as a bit vector.

Problem: Consider $S = \{1, 2, 3, 4\}$ and proposed cover $\{\{1, 3\}, \{1, 4\}, \{1\}\}$.

INDEPENDENT SET problem: Given a graph, is there a set of vertices of size k with none adjacent to each other?

Reduction from 3SAT to INDEPENDENT SET (LP pg 326–327.)

Suppose we have an instance of 3SAT, $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$. WOLOG, suppose each clause has exactly three literals. Construct an instance of INDEPENDENT SET, (G, K) where $K = m$ and $G = (V, E)$ such that

- There is a vertex in V for each literal occurrence (or clause position) $c_{i,j}$.
- $(c_{i,j}, c_{x,y}) \in E$ if either
  - $i = x$ (they are positions in the same clause; this makes a triangle of vertices), or
  - the literals $c_{i,j}$ and $c_{x,y}$ are negations of each other.

Suppose an independent set of size K exists in G. It must include exactly one vertex in each triangle. Make a truth assignment that makes each literal in the set true. Suppose a satisfying truth assignment exists. Then for each triangle, pick one vertex corresponding to a true literal.

Proof that LONGEST CYCLE is $\mathcal{NP}$-Complete

Proof. *[LONGEST CYCLE is $\mathcal{NP}$.] Suppose $(G = (V, E), K)$ is an instance of LONGEST CYCLE and p is a path that is a proposed cycle of length K. An algorithm to check that p is consistent with E, has no repeated vertices, and has length at least K, is polynomial with any reasonable representation of G. Moreover, since p is no larger than the representation of G, it is succinct.*
[LONGEST CYCLE is $\mathcal{NP}$-hard.] Suppose $(G = (V, E))$ is an instance of HAMILTON CYCLE. Then make an instance of LONGEST CYCLE by letting $K = |V|$, which obviously can be done in polynomial time.
Since $K = |V|$, any cycle of length (at least) K must be a Hamilton cycle, and any Hamilton cycle must have length K.
Therefore LONGEST CYCLE is $\mathcal{NP}$-complete. $\square$

Proof that SUBGRAPH ISOMORPHISM is $\mathcal{NP}$-Complete

Proof. *[SUBGRAPH ISOMORPHISM is $\mathcal{NP}$.] Suppose* $(G_1 = (V_1, E_1), G_2 = (V_2, E_2))$ *is an instance of* SUBGRAPH ISOMORPHISM *and* f *is a function* $V_1 \to V_2$ *(expressed as a list of pairs where* $(v_{1,a}, v_{2,b})$ *indicates* $v_{1,a} \in V_1$*,* $v_{2,b} \in V_2$*, and* $f(v_{1,a}) = v_{2,b}$*) proposed as an isomorphism. An algorithm to check that* f *is a one-to-one function and that for all* $(v_{1,a}, v_{1,b}) \in E_1$*,* $(f(v_{1,a}), f(v_{1,b})) \in E_2$*, is polynomial with any reasonable representation of* G*. Moreover, since* $|f| = O(V_1)$*, it is succinct.*
[SUBGRAPH ISOMORPHISM is $\mathcal{NP}$-hard.] Suppose $(H = (W, F))$ *is an instance of* HAMILTON CYCLE*. Then construct a graph* $G = (V, E)$ *that such that* $|V| = |W|$ *and* $E = \{(w_1, w_2), (w_2, w_3), \ldots (w_{|V|}, w_1)\}$ *An algorithm to construct this graph takes* $O(V)$ *time.*
Note that E *has only those edges that make a Hamiltonian cycle. Thus* G *is isomorphic to a subgraph of* H *iff* H *has a Hamiltonian cycle.*
Therefore SUBGRAPH ISOMORPHISM *is* $\mathcal{NP}$*-complete.* $\square$