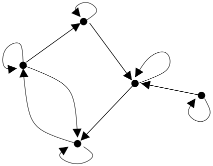
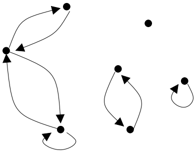
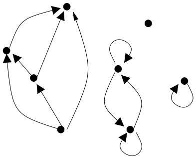
Chapter 5 roadmap:

- Introduction to relations (last week Friday)
- Properties of relations (Monday and Wednesday)
- Closures (**Today**)
- Partial and total order relations (next week Monday)

Today:

- Review of relation properties
- An arithmetic on relations
- Computing whether a function is transitive
- Transitive closure
- Other closures

A **relation** from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A **relation** on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The **image** of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The **image** of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The **inverse** of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The **composition** of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The **identity** relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	$=$

	Reflexivity	**Symmetry**	**Transitivity**
Informal	Everything is related to itself	All pairs are mutual	Anything reachable by two hops is reachable by one hop
Formal	$\forall\ x \in X,\ (x,x) \in R$	$\forall\ x,y \in X,\ (x,y) \in R \to (y,x) \in R$ OR $\forall (x,y) \in R,\ (y,x) \in R$	$\forall x,y,z \in X,$ $(x,y),(y,z) \in R \to (x,z) \in R$ OR $\forall (x,y),(y,z) \in R, (x,z) \in R$
Visual			
Examples	$\subseteq$, $\leq$, $\geq$, $\equiv$, i, isAquaintedWith, waterVerticallyAligned	$\equiv$, isOppositeOf, isOnSameRiver, isAquaintedWith	$<$, $\leq$, $>$, $\geq$, $\subseteq$, isTallerThan, isAncestorOf, isWestOf

The identity relation is a ________________.
noun

Reflexivity is a ________________ that ____________________________________.
noun phrase

Composition is an ________________ on ________________.
noun plural noun

Transitivity is a ________________ that ____________________________________.
noun phrase

Operators	$x + y$	$p \vee q$	$A \cup B$
	$-x$	$\sim p$	$\overline{A}$
Distribution	$x \cdot (y + z)$	$p \wedge (q \vee r)$	$A \cap (B \cup C)$
	$= x \cdot y + x \cdot z$	$\equiv (p \wedge q) \vee (p \wedge r)$	$= (A \cap B) \cup (A \cap C)$
Identity	$x + 0 = x$	$p \wedge T \equiv p$	$A \cup \emptyset = A$
	$x \cdot 1 = x$	$p \vee F \equiv p$	$A \cap \mathcal{U} = A$

$S \circ R$

R^{-1}

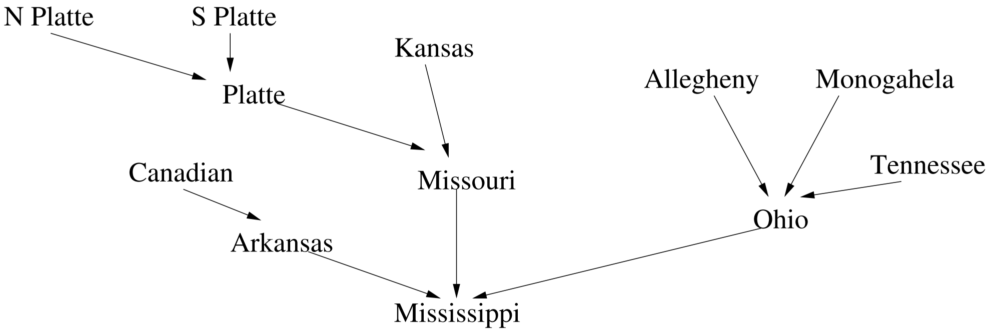
$i_X \circ R = R$

$R^2 = R \circ R$

R	is one less than	eats	is parent of
R^2	is two less than	eats something that eats	is grandparent of
R^3	is three less than	eats something that eats something that eats	is great grandparent of
???	$<$	gets nutrients from	is ancestor of

Domain	First relation	Second relation
Rivers	*flows into* The Platte flows into the Missouri, and the Missouri flows into the Mississippi.	*is tributary to* The Platte is a tributary to the Missouri; both the Platte and the Missouri are tributaries to the Mississippi.
People	*is parent of* Bill is Jane's parent; Jane is Leroy's parent	*is ancestor of* Bill is Jane's ancestor; Leroy has both Jane and Bill as ancestors.

Domain	First relation	Second relation
Animals	*eats*	*derives nutrients from*
	Rabbit eats clover; coyote eats rabbit.	Coyote derives nutrients from rabbit; rabbit derives nutrients from clover; both coyote and rabbit ultimately derive nutrients from clover.
$\mathbb{Z}$	*is one less than*	$<$
	2 is one less than 3; 3 is one less than 4	$2 < 3$; $3 < 4$; $2 < 4$.


N Platte
S Platte
Kansas
Platte
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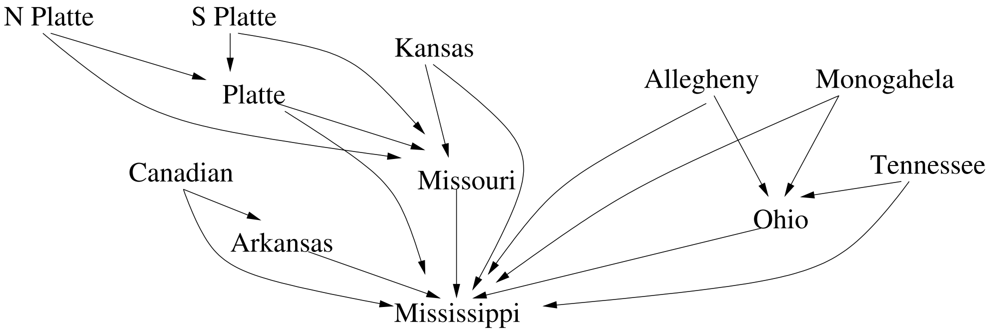
Arkansas
Mississippi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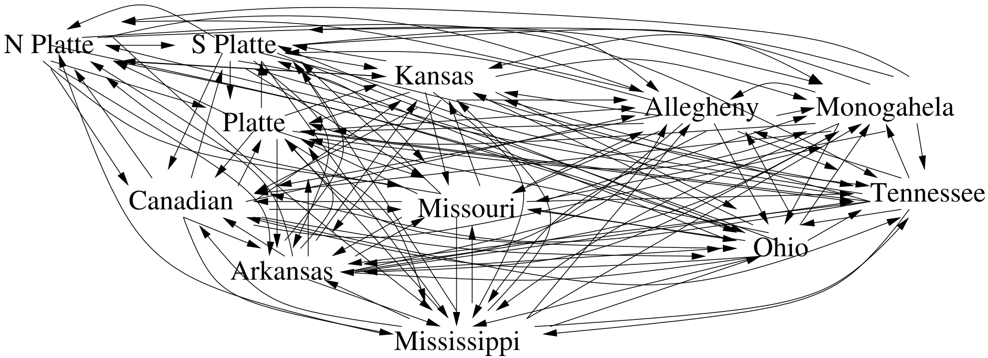
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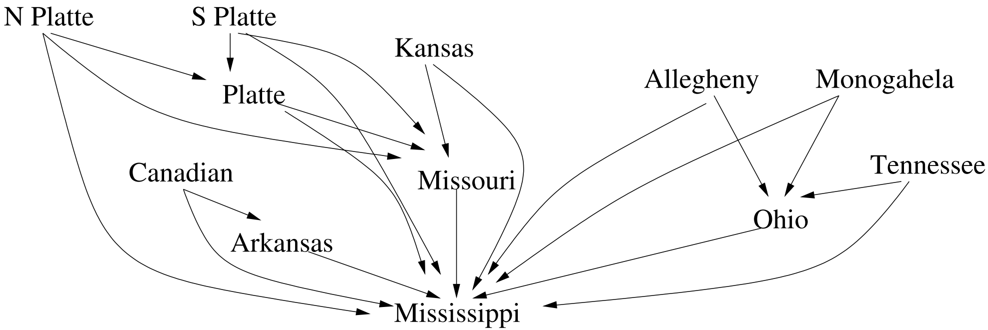

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Canadian
Missouri
Ohio
Arkansas
Mississippi

If R is a relation on X, then R^T is the **transitive closure** of R if

- R^T is transitive
- $R \subseteq R^T$
- If S is a transitive relation such that $R \subseteq S$, then $R^T \subseteq S$

Which of the following expresses a transitive closure?

- My friends are my friends, an no one else.
- Any friend of my friend is also my friend.
- Any friend of my friends' friends is also my friend.
- My friends are my friends, and so are my friends's friends, and so are my friends' friends' friends, ans so on forever.

Let R be a relation and let T be the transitive closure of R. What, then, do you know to be true? Select all that apply.

- R is transitive
- T is a proposition
- T is a relation
- T is transitive
- T is a powerset
- $R \subseteq T$
- $T \subseteq R$

Theorem 5.13 *If R is a relation on a set A, then*

$$R^\infty = \bigcup_{i=1}^{\infty} R^i = \{(x, y) \mid \exists\ i \in \mathbb{N} \text{ such that } (x, y) \in R^i\}$$

is the transitive closure of R.

Proof. *Suppose R is a relation on a set A.*
Suppose $a, b, c \in A$, $(a, b), (b, c) \in R^\infty$. By the definition of R^∞, there exist $i, j \in \mathbb{N}$ such that $(a, b) \in R^i$ and $(b, c) \in R^j$. By the definition of relation composition and Exercise 5.7.4, $(a, c) \in R^j \circ R^i = R^{i+j}$. $R^{i+j} \subseteq R^\infty$ by the definition of R^∞. By the definition of subset, $(a, c) \in R^\infty$. Hence, R^∞ is transitive by definition.
Suppose $a, b \in A$ and $(a, b) \in R$. By the definition of R^∞ (taking $i = 1$), $(a, b) \in R^\infty$, and so $R \subseteq R^\infty$, by definition of subset.
Suppose S is a transitive relation on A and $R \subseteq S$. Further suppose $(a, b) \in R^\infty$. Then, by definition of R^∞, there exists $i \in \mathbb{N}$ such that $(a, b) \in R^i$. By Lemma 5.14, $(a, b) \in S$. Hence $R^\infty \subseteq S$ by definition of subset.
Therefore, R^∞ is the transitive closure of R. □

Other closures:

Ex 5.4.9 $R \cup i_A$ is the reflexive closure of R

Ex 5.4.10 $R \cup R^{-1}$ is the symmetric closure of R. (HW)

Ex 5.4.9 $R \cup i_A$ is the reflexive closure of R

Proof. *Suppose R is a relation on A.*

[$R \cup i_A$ is reflexive:] Suppose $a \in A$. $(a, a) \in i_A$ by definition of identity relation. $(a, a) \in R \cup i_A$ by definition of union. Hence $R \cup i_A$ is reflexive by definition.

[$R \subseteq R \cup i_A$:] Suppose $(a, b) \in R$. Then $(a, b) \in R \cup i_A$ by definition of uniion. Hence $R \subseteq R \cup i_A$. (Alternately, we could have cited Exercise 4.2.1.)

[$R \cup i_A$ is the smallest such relation:] Suppose S is a reflexive relation such that $R \subseteq S$. Suppose further $(a, b) \in R \cup i_A$. By definition of union, $(a, b) \in R$ or $(a, b) \in i_A$.

Case 1: *Suppose $(a, b) \in R$. Then $(a, b) \in S$ by definition of subset (since we supposed $R \subseteq S$).*

Case 2: *Suppose $(a, b) \in i_A$. Then, by definition of identity relation, $a = b$. $(a, a) \in S$ by definition of reflexive (since we suppose S is reflexive). $(a, b) \in S$ by substitution.*

Either way, $(a, b) \in S$ and hence $R \cup i_A \subseteq S$ by definition of subset.

Therefore, $R \cup i_A$ is the reflexive closure of R. □

For next time:

Do Exercises 5.4.(1, 2, 3, 4, 5, 9, 10)

Read 5.5

Take quiz