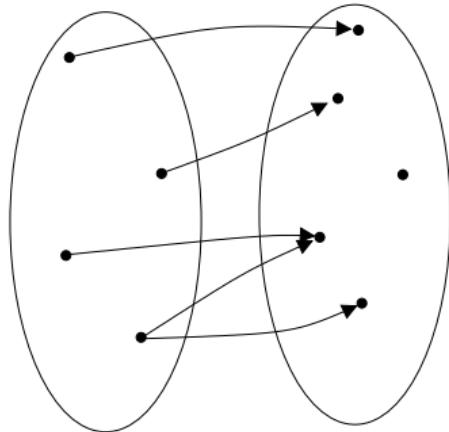


Chapter 6 outline:

- ▶ Introduction, function equality, and dictionaries (last week Wednesday)
- ▶ Image and inverse images (last week Friday)
- ▶ Function properties and composition (**Today**)
- ▶ Reducing and pipelining (Wednesday)
- ▶ Cardinality (Friday)
- ▶ Countability and practice quiz (next week Monday)
- ▶ Review (next week Wednesday)
- ▶ Test 3, on Ch 5 & 6 (next week Friay, Nov 21)

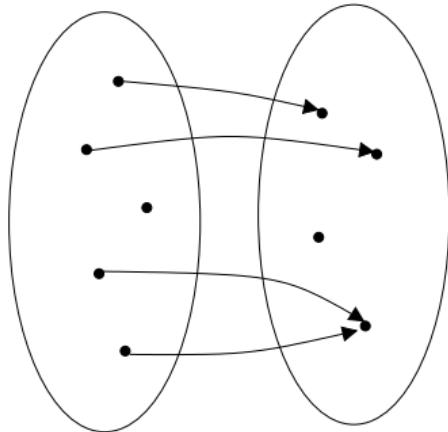
Today:

- ▶ Definition of one-to-one and onto, plus proofs
- ▶ Inverse functions
- ▶ Definition of function composition, plus proofs



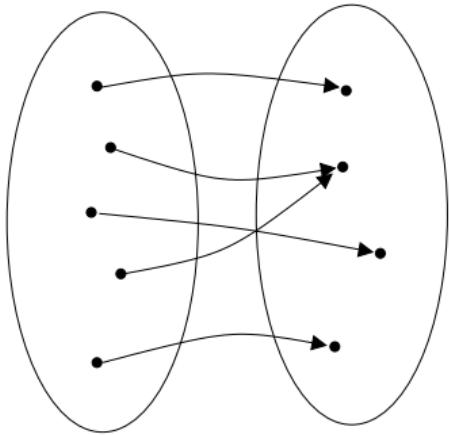
Not a function.

(There's a domain element that is related to two things.)



Not a function.

(There's a domain element that is not related to anything.)



Onto (Surjection)

Everything in the codomain is hit.

$f : X \rightarrow Y$ is onto if $\forall y \in Y$,
 $\exists x \in X \mid f(x) = y$.

Analytic use:

f is onto.

$y \in Y$.

Hence $\exists x \in X$ such that $f(x) = y$.

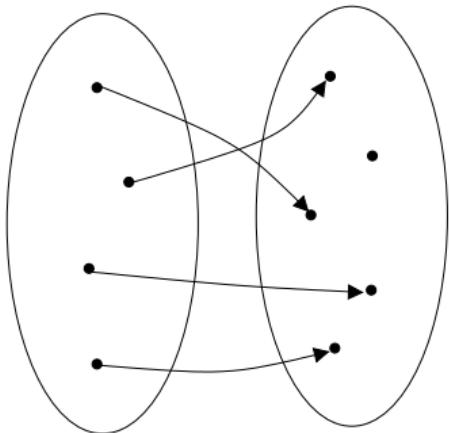
Synthetic use:

Suppose $y \in Y$.

⋮

(Somehow find x such that $f(x) = y$.)

Therefore f is onto.



One-to-one (Injection)

Domain elements don't share.

f is one-to-one if $\forall x_1, x_2 \in X$,
if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Analytic use:

f is one-to-one.

$$f(x_1) = f(x_2).$$

Hence $x_1 = x_2$.

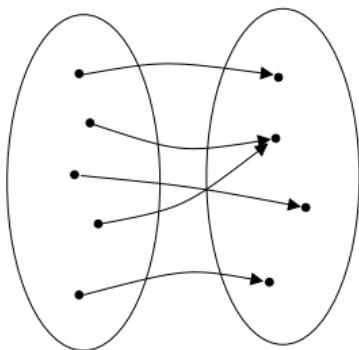
Synthetic use:

Suppose $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$.

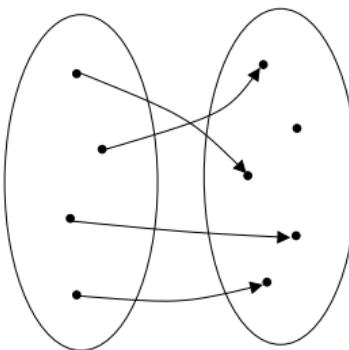
⋮

(Somehow show $x_1 = x_2$.)

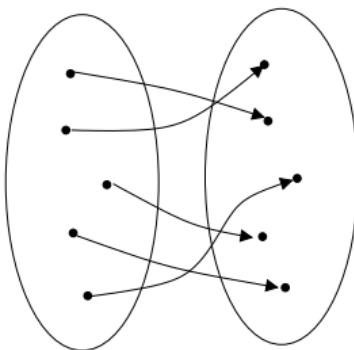
Therefore f is one-to-one.



Onto
(not one-to-one)
 $|X| \geq |Y|$

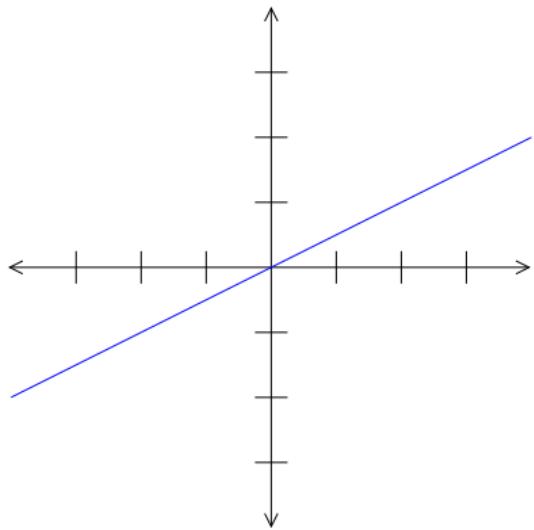


One-to-one
(not onto)
 $|X| \leq |Y|$



Both onto and one-to-one
 $|X| = |Y|$

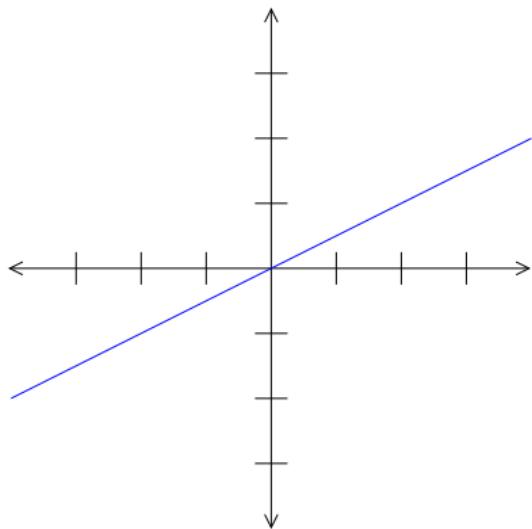
Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?



Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

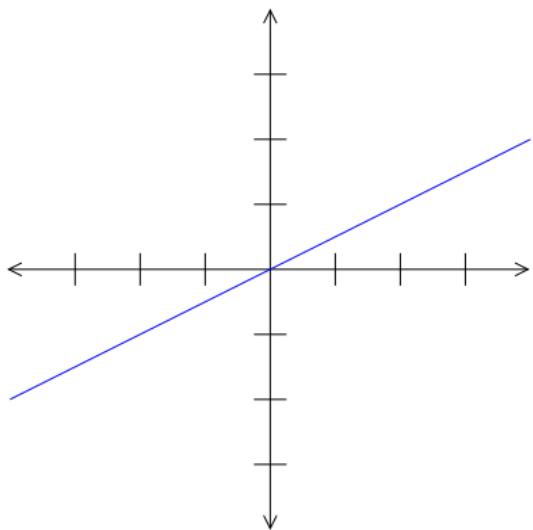
Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how f is defined,



Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how f is defined,



$$\begin{aligned}\frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2\end{aligned}$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

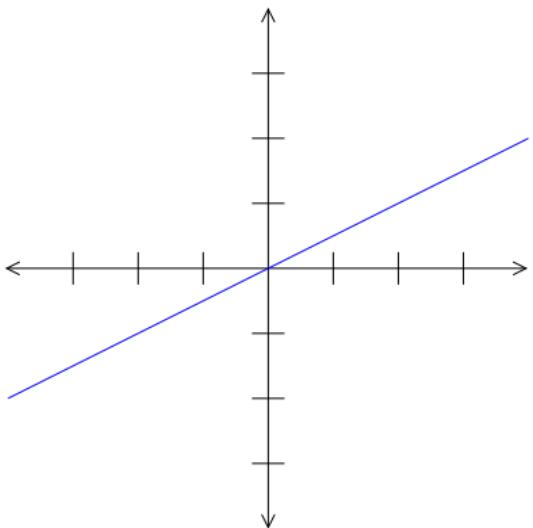
f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how f is defined,

$$\begin{aligned}\frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2\end{aligned}$$

Therefore f is one-to-one by definition. \square

f is onto.



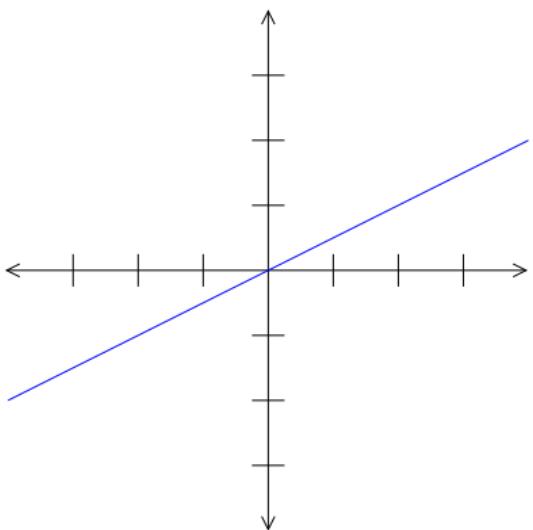
Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how f is defined,

$$\begin{aligned}\frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2\end{aligned}$$

Therefore f is one-to-one by definition. \square



f is onto.

Proof. Suppose $y \in \mathbb{R}$. [Want x such that $f(x) = y$.]

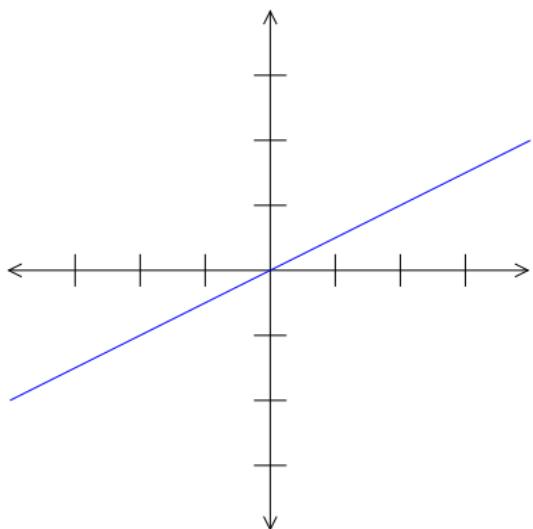
Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is f one-to-one? Is it onto?

f is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how f is defined,

$$\begin{aligned}\frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2\end{aligned}$$

Therefore f is one-to-one by definition. \square



f is onto.

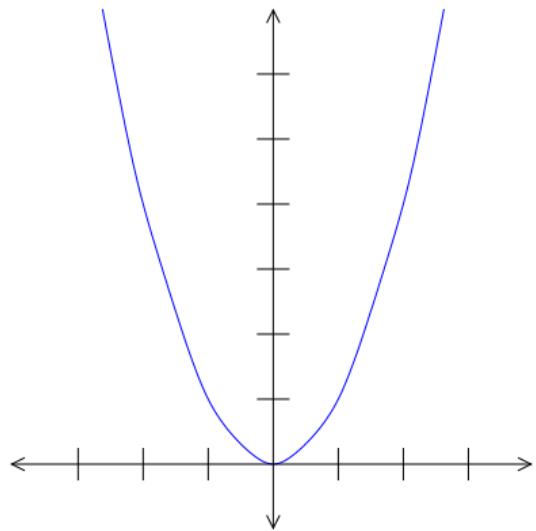
Proof. Suppose $y \in \mathbb{R}$. [Want x such that $f(x) = y$.]

Let $x = 2y$. Then

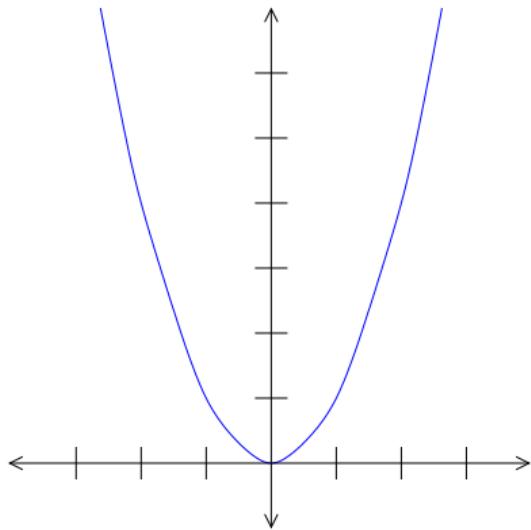
$$\begin{aligned}f(x) &= \frac{2y}{2} \\ &= y\end{aligned}$$

Therefore f is onto by definition \square

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = x^2$. Is f one-to-one? Is it onto?



Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = x^2$. Is f one-to-one? Is it onto?



f is not one-to-one.

$$f(2) = 2^2 = 4$$

$$f(-2) = (-2)^2 = 4$$

f is not onto.

Let $y = -1$.

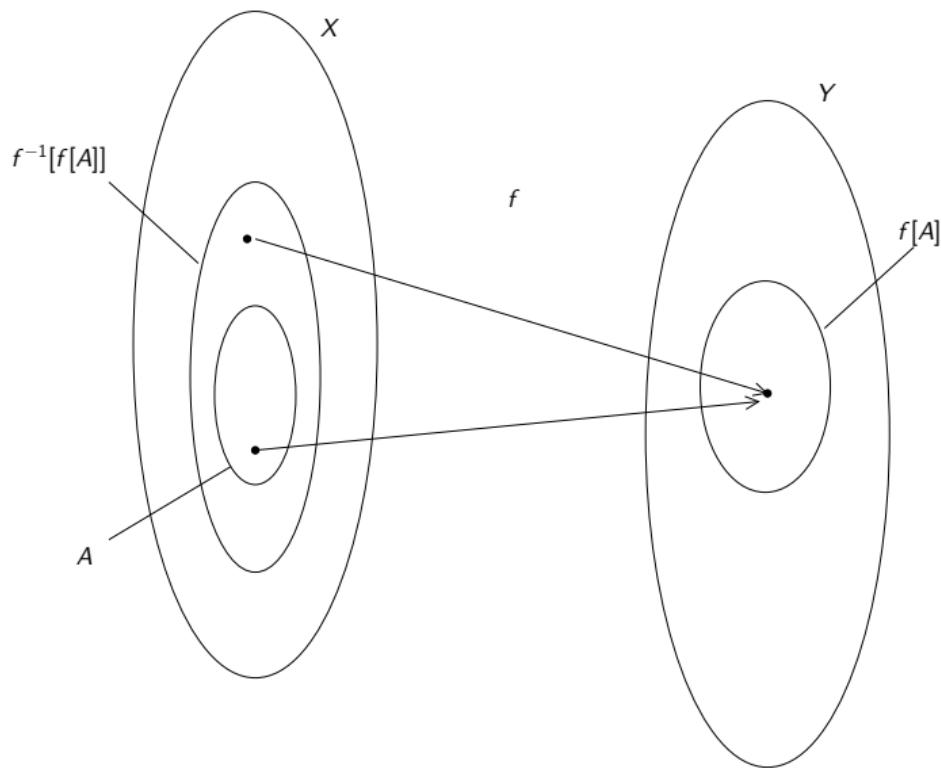
$\nexists x \in \mathbb{R}$ such that $f(x) = -1$.

Ex 6.3.4. If $A \subseteq X$ and f is one-to-one, then $f^{-1}[f[A]] \subseteq A$.

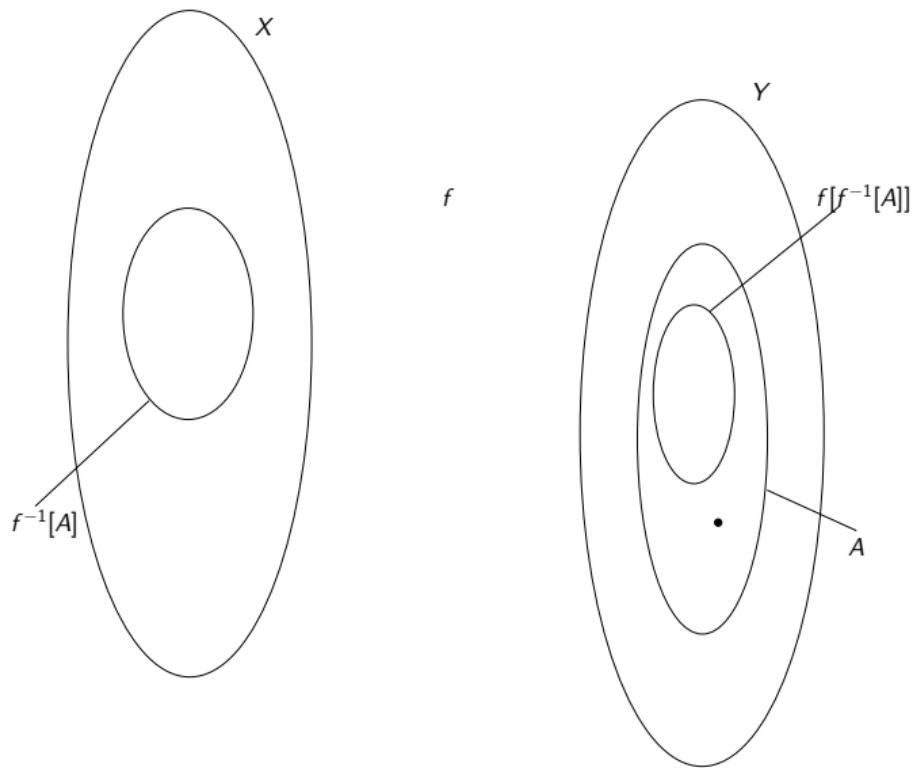
(Ex 6.2.9 was, Prove $A \subseteq f^{-1}[f[A]]$, and Ex 6.2.10 was, Find a counterexample for $A = f^{-1}[f[A]]$.)

Ex 6.3.4. If $A \subseteq X$ and f is one-to-one, then $f^{-1}[f[A]] \subseteq A$.

(Ex 6.2.9 was, Prove $A \subseteq f^{-1}[f[A]]$, and Ex 6.2.10 was, Find a counterexample for $A = f^{-1}[f[A]]$.)



Ex 6.3.5. If $A \subseteq Y$ and f is onto, then $A \subseteq f[f^{-1}[A]]$.



Inverse relation: $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$

Since a function is a relation, a function has an inverse, but we don't know that the inverse of a function is a function.

If $f : X \rightarrow Y$ is a **one-to-one correspondence**, then

$$f^{-1} : Y \rightarrow X = \{(y, x) \in Y \times X \mid f(x) = y\}$$

is the *inverse function* of f .

Theorem 6.8 *If $f : X \rightarrow Y$ is a one-to-one correspondence, then $f^{-1} : Y \rightarrow X$ is well defined.*

Proof. Suppose $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Hence $(y, x) \in f^{-1}$ or $f^{-1}(y) = x$.

Further suppose $(y, x_1), (y, x_2) \in f^{-1}$ (*That is, suppose that both $f^{-1}(y) = x_1$ and $f^{-1}(y) = x_2$.*) Then $f(x_1) = y$ and $f(x_2) = y$. Since f is one-to-one, $x_1 = x_2$.

Therefore, by definition of function, f^{-1} is well defined. \square

Relation composition: If R is a relation from X to Y and S is a relation from Y to Z , then $S \circ R$ is the relation from X to Z defined as

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

Function composition: If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then $g \circ f : X \rightarrow Z$ is defined as

$$g \circ f = \{(x, z) \in X \times Z \mid z = g(f(x))\}$$

Theorem 6.9 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, then $g \circ f : X \rightarrow Z$ is well defined.*

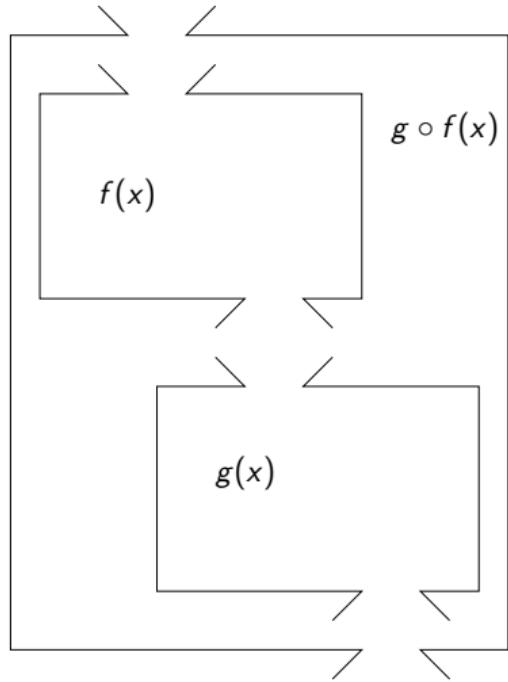
Proof. Suppose $x \in X$. Since f is a function, there exists a $y \in Y$ such that $f(x) = y$. Since g is a function, there exists a $z \in Z$ such that $g(y) = z$. By definition of composition, $(x, z) \in g \circ f$, or $g \circ f(x) = z$.

Next suppose $(x, z_1), (x, z_2) \in g \circ f$, or $g \circ f(x) = z_1$ and $g \circ f(x) = z_2$. By definition of composition, there exist y_1, y_2 such that $f(x) = y_1$, $f(x) = y_2$, $g(y_1) = z_1$, and $g(y_2) = z_2$. Since f is a function, $y_1 = y_2$. Since g is a function, $z_1 = z_2$.

Therefore, by definition of function, $g \circ f$ is well defined. \square

Function composition: If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then $g \circ f : X \rightarrow Z$ is defined as

$$g \circ f = \{(x, z) \in X \times Z \mid x = g(f(x))\}$$



Let $f(x) = 3x$

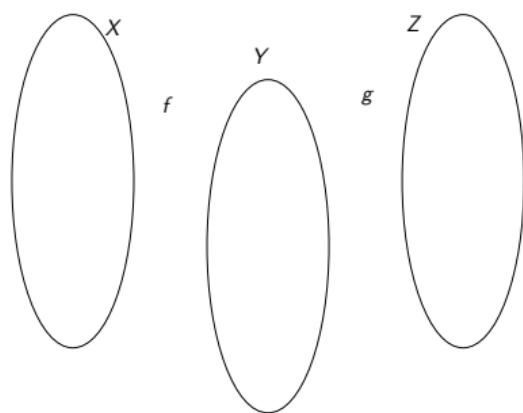
Let $g(x) = x + 7$

Then

$$\begin{aligned} g \circ f(x) &= f(x) + 7 \\ &= 3x + 7 \end{aligned}$$

Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

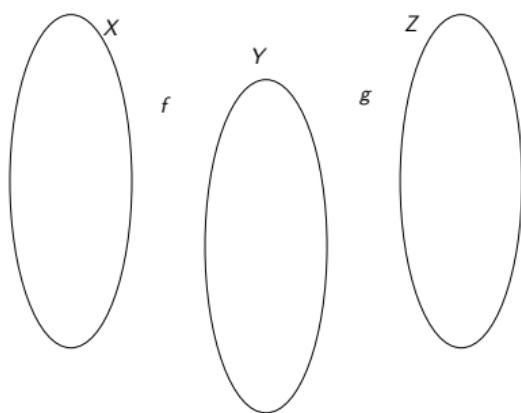
Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.



Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

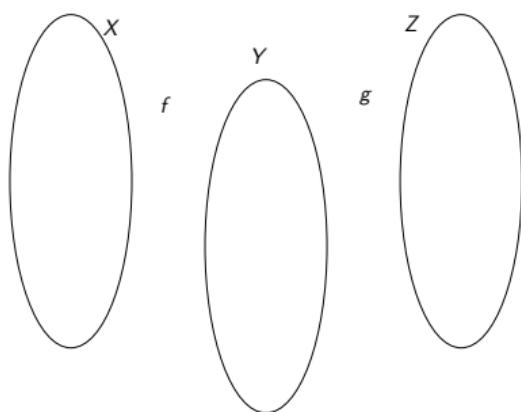
[Now, we want to prove “ontoness.” Of which function?



Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

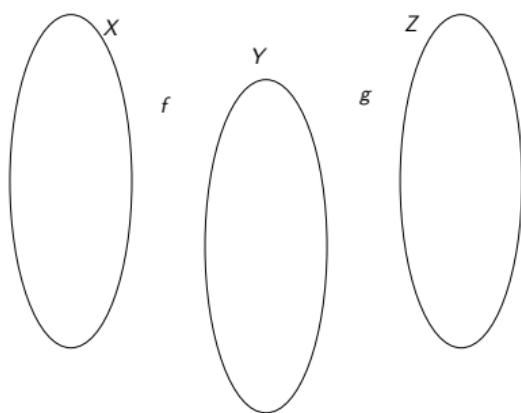
[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness?



Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$?

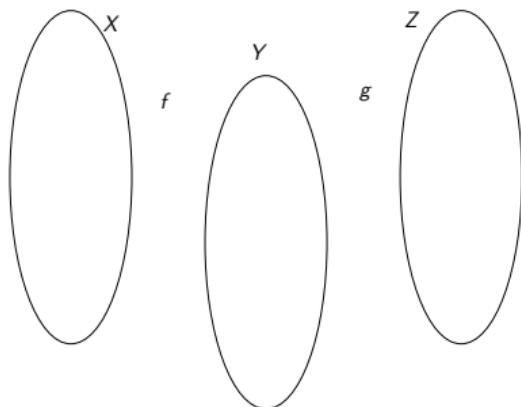


Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$? Z .]

Further suppose $z \in Z$. [We need to come up with something in the domain of $g \circ f$ that hits z . The domain is X . We will use the fact that f and g are both onto.]

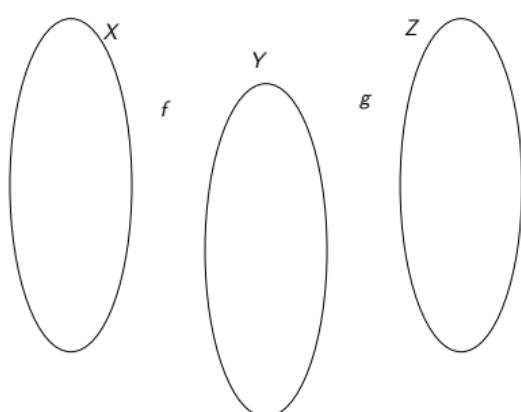


Ex 6.4.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$? Z .]

Further suppose $z \in Z$. [We need to come up with something in the domain of $g \circ f$ that hits z . The domain is X . We will use the fact that f and g are both onto.]

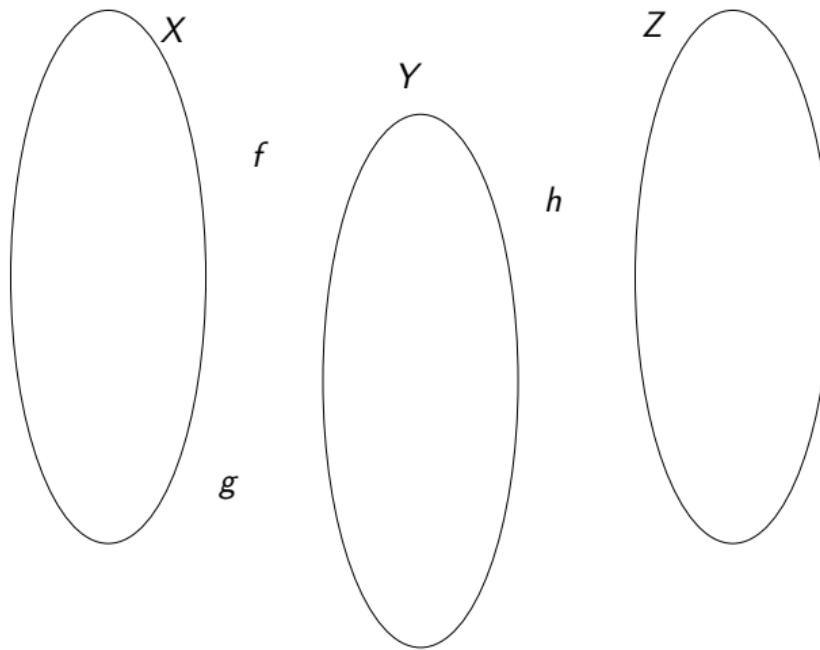


By definition of onto, there exists $y \in Y$ such that $g(y) = z$. Similarly there exists $x \in X$ such that $f(x) = y$. Now,

$$\begin{aligned} g \circ f(x) &= g(f(x)) && \text{by definition of function composition} \\ &= g(y) && \text{by substitution} \\ &= z && \text{by substitution} \end{aligned}$$

Therefore $g \circ f$ is onto by definition. \square

Ex 6.4.8. If $f : X \rightarrow Y$, $g : X \rightarrow Y$ and $h : Y \rightarrow Z$, h is one-to-one, and $h \circ f = h \circ g$, then $f = g$.



For next time:

Do Exercises 6.3.(2,3,6) and 6.4.(1,5,6)

*Read Section 6.5 (**new**)*

Take quiz