Chapter 6 in context:

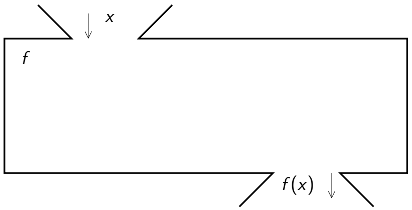
- Chapter 5 Relations: Builds on proofs about sets
- Chapter 6 Function: Builds on proofs about relations
- Chapter 7 Self Reference: Focuses on recursive thinking

Chapter 6 outline:

- Introduction, function equality, and dictionaries (**Today**)
- Image and inverse images (Friday)
- Function properties and composition (next week Monday)
- Reducing and pipelining (next week Wednesday)
- Cardinality (next week Friday)
- Countability (week-after Monday, Nov 17)
- Review (week-after Wednesday, Nov 19)
- Test 3, on Ch 5 & 6 (week-after Friday, Nov 21)

A function is. . .

- a parameterized expression.
- a named piece of code that can be invoked many times in different contexts.
- an extension to the programming language.
- an abstract machine.
- a *value*.



Cross out the term/concept that was **not** used in the reading for today as a way to think about functions

A kind of machine

A topological sort

A mapping between two collections

A kind of relation

For the function $f : X \to Y$, $X$ is the ____________________ and $Y$ is the

____________________.

function

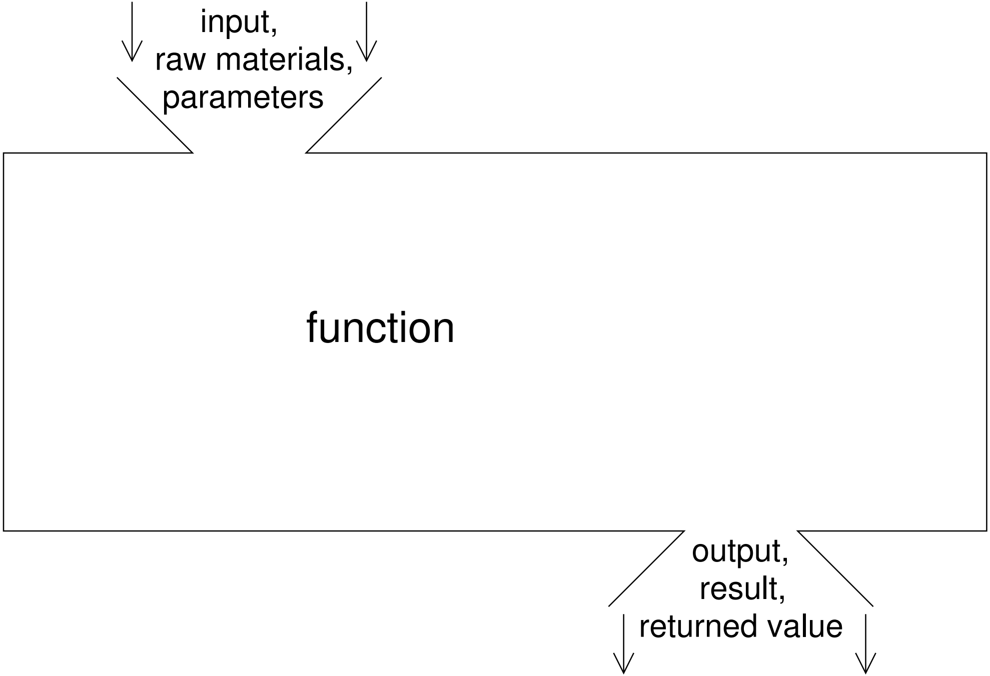
constant

domain

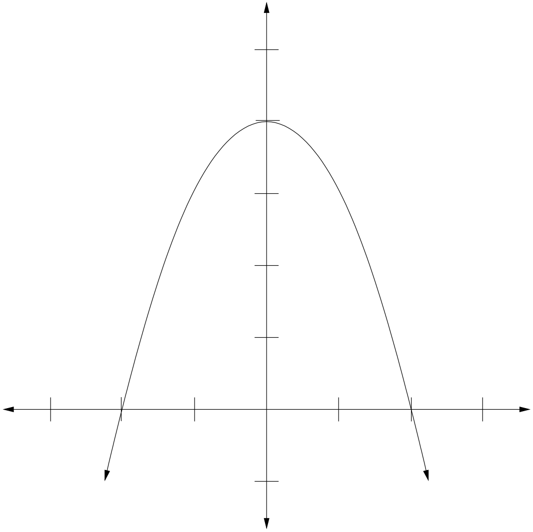
codomain

first-class value

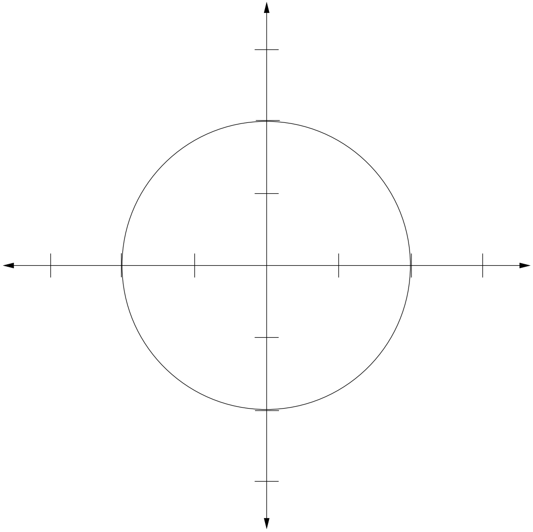
relation


input,
raw materials,
parameters
function
output,
result,
returned value

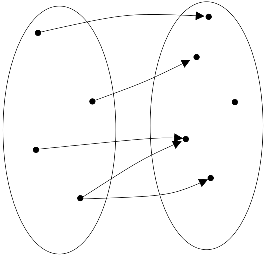
| | |
|---|---|
| Alice | x3498 |
| Bob | x4472 |
| Carol | x5392 |
| Dave | x9955 |
| Eve | x2533 |
| Fred | x9448 |
| Georgia | x3684 |
| Herb | x8401 |



$y = 4 - x^2$

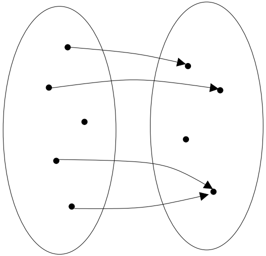


$y^2 = 4 - x^2$



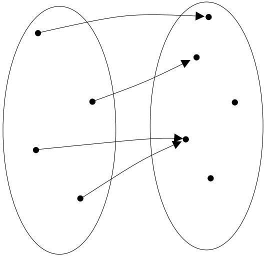
**Not a function.**

(There's a domain element that is related to two things.)



**Not a function.**

(There's a domain element that is not related to anything.)



**A function.**

(It's OK that two domain elements are related to the same thing and one codomain element has nothing related to it.)

## Definition of function

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal: $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \quad \exists\, y \in Y \mid (x, y) \in f \qquad \textbf{existence of } y$$

$$\wedge \quad \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2 \qquad \textbf{uniqueness of } y$$

## Change of notation

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal (relation notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \quad \exists\, y \in Y \mid (x, y) \in f \qquad \textbf{existence of } y$$

$$\wedge \quad \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2 \qquad \textbf{uniqueness of } y$$

Formal (function notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \quad \exists\, y \in Y \mid f(x) = y \qquad \textbf{existence of } y$$

$$\wedge \quad \forall\, y_1, y_2 \in Y, (f(x) = y_1 \wedge f(x) = y_2) \rightarrow y_1 = y_2 \qquad \textbf{uniqueness of } y$$

We call $X$ the *domain* and $Y$ the *codomain* of $f$.

**Definition of function equality.** Let $f, g : X \to Y$

Old definition: functions are sets.

$$f = g \text{ if } \forall\, f \subseteq g \ \wedge g \subseteq f$$

New definition: based on function notation.

$$f = g \ \text{ if } \ \forall\, x \in X, f(x) = g(x)$$

Function equality: $f = g$ if $\forall\, x \in X, f(x) = g(x)$

Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x \cdot (x - 1) - 6$ and $g(x) = (x - 3)(x + 2)$.

Prove $f = g$.

The old and new definitions of function equality are equivalent.

**Ex 6.1.1.** $(\forall\, x \in X, f(x) = g(x)) \quad \text{iff} \quad (f \subseteq g \wedge g \subseteq f)$.

The old and new definitions of function equality are equivalent.

**Ex 6.1.1.** $(\forall\, x \in X, f(x) = g(x))$ iff $(f \subseteq g \wedge g \subseteq f)$.

**Proof.** First, suppose $\forall\, x \in X, f(x) = g(x)$, that is, $f = g$ by definition of function equality. Further suppose $(x, y) \in f$. By function notation, $f(x) = y$. By supposition and substitution, $g(x) = y$. By relation notation, $(x, y) \in g$. Finally, $f \subseteq g$ by definition of subset.

Similarly $g \subseteq f$, and therefore $f = g$ by definition of set equality.

Conversely, suppose $f \subseteq g \wedge g \subseteq f$, that is, $f = g$ by definition of set equality. Further suppose $x \in X$.

Let $y = f(x)$. Note that this $y \in Y$ must exist by definition of function. By relation notation, $(x, y) \in f$.

By definition of subset [or set equality], $(x, y) \in g$. In function notation, that is $g(x) = y$, and so $f(x) = g(x)$ by substitution. Therefore $f = g$ by definition of function equality. □

**For next time:**

*Do Exercises 6.1.(2,3,7,8,9,10,11,12).*

*Exercises 2 and 3 are function-equality proofs. The other exercises are programming problems.*

*Read Section 6.2.*

*Take quiz*