

Chapter 7 outline:

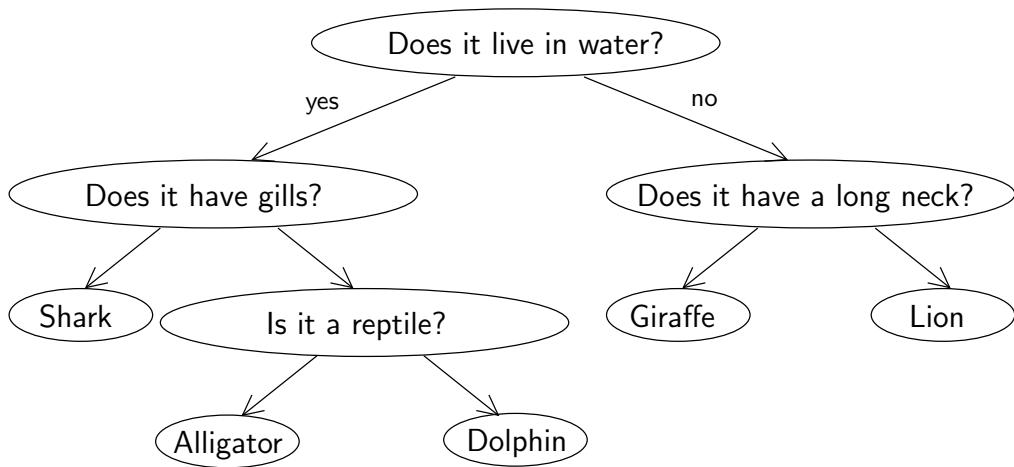
- ▶ Recursively-defined sets (last week Monday)
- ▶ Trees (Monday)
- ▶ Structural induction (**Today**)
- ▶ Mathematical induction (Friday)
- ▶ Loops (next week Monday)
- ▶ Loop invariants (next week Wednesday)
- ▶ Review for final exam (next week Friday)
- ▶ Final exam (Tues, Dec 16, 1:30pm)

Previously we saw

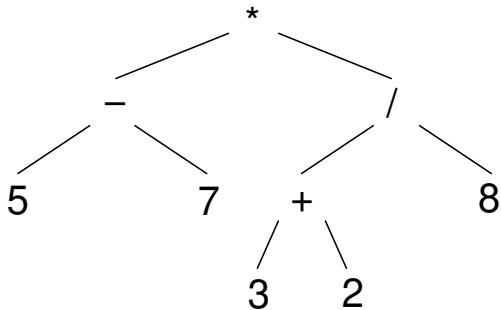
- ▶ A recursive definition of whole numbers
- ▶ A recursive definition of trees, particularly *full binary trees*; a full binary tree is either
 - ▶ a leaf, or
 - ▶ an internal node together with two children which are full binary trees.

Today we see

- ▶ Self-referential proofs



Expression trees:

$$\begin{aligned} \text{Expression} &\rightarrow \text{Variable} \mid \text{Constant} \\ &\quad \mid \text{Expression Operator Expression} \end{aligned}$$
$$\text{Operator} \rightarrow + \mid - \mid * \mid /$$


Tree

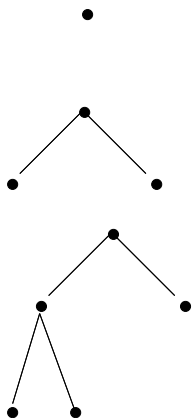
Nodes

Links

Tree

Nodes

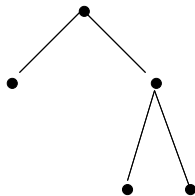
Links



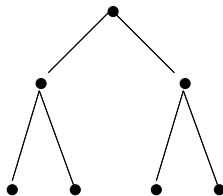
1 0

3 2

5 4



5 4



7 6

While building bigger trees from smaller trees, *the number of nodes is (and remains) one more than the number of links.* (Invariant)

Theorem 7.1 *For any full binary tree T , $\text{nodes}(T) = \text{links}(T) + 1$.*

Let \mathcal{T} be the set of full binary trees. Then, we're saying

$$\forall T \in \mathcal{T}, \text{nodes}(T) = \text{links}(T) + 1$$

Theorem 7.2 For any full binary tree T , $\text{nodes}(T) = \text{links}(T) + 1$.

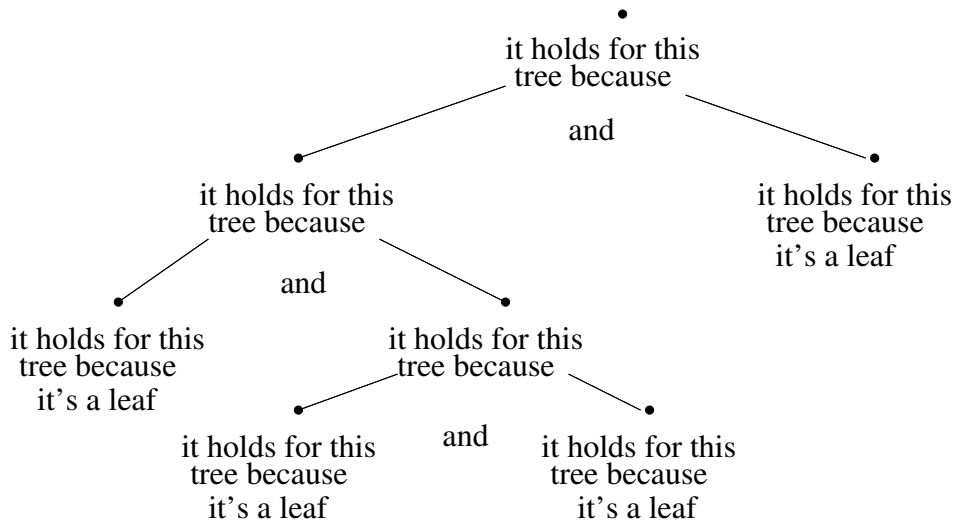
Proof. Suppose $T \in \mathcal{T}$. [What is a tree? the definition says it's either a leaf or an internal with two subtrees. We can use division into cases.]

Case 1. Suppose T is a leaf. Then, by how nodes and links are defined, $\text{nodes}(T) = 1$ and $\text{links}(T) = 0$. Hence $\text{nodes}(T) = \text{links}(T) + 1$.

Case 2. Suppose T is an internal node with links to subtrees T_1 and T_2 . Moreover, by how nodes and links are defined, $\text{links}(T) = \text{links}(T_1) + \text{links}(T_2) + 2$. Then,

$$\begin{aligned} \text{nodes}(T) &= 1 + \text{nodes}(T_1) + \text{nodes}(T_2) && \text{by the definition of nodes} \\ &= 1 + \text{links}(T_1) + 1 + \text{links}(T_2) + 1 && \text{by Theorem 7.2} \\ &= \text{links}(T_1) + \text{links}(T_2) + 2 + 1 && \text{by algebra} \\ &= \text{links}(T) + 1 && \text{by substitution} \end{aligned}$$

Either way, $\text{nodes}(T) = \text{links}(T) + 1$. \square



Theorem 7.2 For any full binary tree T , $\text{nodes}(T) = \text{links}(T) + 1$.

Proof. Suppose $T \in \mathcal{T}$.

Base case. Suppose T is a leaf. Then, by how nodes and links are defined, $\text{nodes}(T) = 1$ and $\text{links}(T) = 0$. Hence $\text{nodes}(T) = \text{links}(T) + 1$.

Inductive case Suppose T is an internal node with links to subtrees T_1 and T_2 such that $\text{nodes}(T_1) = \text{links}(T_1) + 1$ and $\text{nodes}(T_2) = \text{links}(T_2) + 1$. Moreover, by how nodes and links are defined, $\text{links}(T) = \text{links}(T_1) + \text{links}(T_2) + 2$. Then,

$$\begin{aligned} \text{nodes}(T) &= 1 + \text{nodes}(T_1) + \text{nodes}(T_2) && \text{by the definition of nodes} \\ &= 1 + \text{links}(T_1) + 1 + \text{links}(T_2) + 1 && \text{by the inductive hypothesis} \\ &= \text{links}(T_1) + \text{links}(T_2) + 2 + 1 && \text{by algebra} \\ &= \text{links}(T) + 1 && \text{by substitution} \end{aligned}$$

Either way, $\text{nodes}(T) = \text{links}(T) + 1$. \square

Let X be a recursively defined set, and let $\{Y, Z\}$ be a partition of X , where Y is defined by a simple set of elements $Y = \{y_1, y_2, \dots\}$ and Z is defined by a recursive rule.

Examples:

- ▶ X is the set of pizzas, $Y = \text{Crusts}$, and
 $Z = \{(top, bot) \mid top \in \text{Toppings} \wedge bot \in X\}$
- ▶ $X = \mathbb{W}$, $Y = \{0\}$, and $Z = \{\text{succ}(n) \mid n \in \mathbb{W}\}$
- ▶ $X = \mathcal{T}$, Y is the set of leaves, and Z is the set of internals with children
 $T_1, T_2 \in \mathcal{T}$.

Let X be a recursively defined set, and let $\{Y, Z\}$ be a partition of X , where Y is defined by a simple set of elements $Y = \{y_1, y_2, \dots\}$ and Z is defined by a recursive rule.

To prove something in the form of $\forall x \in X, I(x)$, do this:

Base case: Suppose $x \in Y$.

\vdots

$I(x)$

Inductive case: Suppose $x \in Z$. *[Using x and the definition of Z , find components $a, b, \dots \in X$.]*

Suppose $I(a), I(b), \dots$ *[The **inductive hypothesis**]*

\vdots

Use the inductive hypothesis

\vdots

$I(x) \square$

7.4.1 For any full binary tree T , $\text{leaves}(T) = \text{internals}(T) + 1$.

Let the *height* of a full binary tree be 1 if the tree is a node by itself (leaf), or 1 more than the maximum height of its two children, if it is an internal node.

7.4.7 For any full binary tree T , $\text{nodes}(T) \leq 2^{\text{height}(T)} - 1$.

Tree

Nodes

Height

Tree

Nodes

Height



1

1



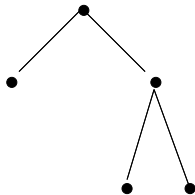
3

2



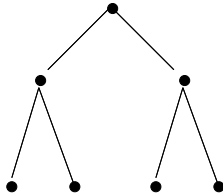
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3



5

3



7

3

For next time:

Do Exercises 7.4.(2,3,5).

Read Section 7.5

Take quiz