Boolean Algebras

First of all, we're going to make a quick change to our notation. Instead of calling the greatest element in a lattice 1 and the least element 0, we'll prefer how some texts have it, calling them $\top$ (pronounced *top*) and $\bot$ (pronounced *bottom*), respectively. It will make more sense to call 30 $\top$ in D_{30}, rather than saying that 30 is 1 in D_{30}

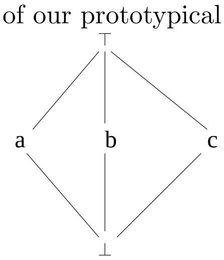
Recall from last time that if A is a bounded lattice and $a \in A$, then the *complement* of a is an element $a' \in A$ such that *complement*

$$a \vee a' = \top \qquad \text{and} \qquad a \wedge a' = \bot$$

This definition, of course, does not imply that such a complement *exists*, nor that it is unique if it exists. We did, however, show the following theorem.

Theorem 1 *If A is a bounded, distributive lattice and $a \in A$, then a', if it exists, is unique.*

Now, we define a bounded lattice A to be *complemented* if for all $a \in A$, a' exists. The following lattice is complemented, since a, b, and c are complements of each other, and $\top$ and $\bot$ are complements of each other. (Notice that this is one of our prototypical non-distributive lattices) *complemented*



I should have made the proof of the following theorem an exercise last time.

Theorem 2 *Complementedness is an isomorphic invariant.*

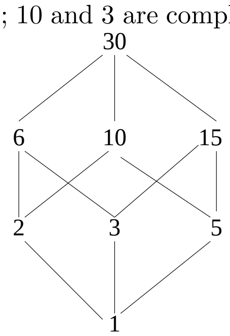
Proof. Suppose $A, \preceq$ and $B, \preceq'$ are isomorphic lattices with isomorphism $f : A \leftarrow B$. Suppose further that A is complemented.

Suppose $b \in B$, and let $a = f^{-1}(b)$. Since A is complemented, there exists $a' \in A$ such that $a \vee a' = \top_A$ and $a \wedge a' = \bot_A$. Then

$$
\begin{aligned}
b \vee f(a') &= f(f^{-1}(b) \vee a') && \text{by your Exercise 3} \\
&= f(a \vee a') && \text{by substitution} \\
&= f(\top_A) && \text{by substitution} \\
&= \top_B && \text{since } \top \text{ and } \bot \text{ match across isomorphisms}
\end{aligned}
$$

The fact that $b \wedge f(a') = \bot_B$ is similar. Hence $f(a')$ is a complement of b, and B is complemented. □

Now the big definition: A *boolean algebra* is a a complemented, distributive lattice. (Notice that being complemented implies being bounded. All finite lattices are bounded, but not all bounded lattices are finite. Nevertheless, we will assume finite boolean algebras for our discussion.) Both our running examples, D_{30} and $\mathscr{P}(2, 3, 5)$, are boolean algebras. Take D_{30}. 30 and 1 are complements; 6 and 5 are complements; 15 and 2 are complements; 10 and 3 are complements (notice a pattern?). *boolean algebra*



Corollary 1 *If A is a boolean algebra, then for all $a \in A$, a has a unique complement.*

Proof. Since boolean algebras are complemented, at least one complement, a', exists. Since boolean algebras are distributive, Theorem 1 tells us that a' is unique. □

Let $\mathbb{B}$ be the set of boolean values, { true, false }. Let $\mathbb{B}_n$ be the set of n-tuples over $\mathbb{B}$. For example, $\mathbb{B}_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$. (We call "false" 0 and "true" 1 and omit the commas and parentheses from tuple notation in order to reduce clutter. "000" would conventionally be written "(false, false, false)". We will also sometimes treat the values as the numbers 0 and 1 rather than the boolean values true and false.) If $a \in \mathbb{B}_n$, then we refer to the components of a as $a_1, \ldots, a_n$. $\mathbb{B}$

Define $\preceq$ so that for $a, b \in \mathbb{B}_n$, $a \preceq b$ if for all i, $1 \leq i \leq n$, $a_i \leq b_i$ (interpreting the components as numbers). It's straightforward to see that $\mathbb{B}_n, \preceq$ is a lattice.

Exercises. (To think about)

1. What are the $\vee$ and $\wedge$ operations on $\mathbb{B}_n$?
2. For $x \in \mathbb{B}_n$, what is the complement of x (if any)?
3. Show that $\mathbb{B}_n$ is a boolean algebra.