Graphs

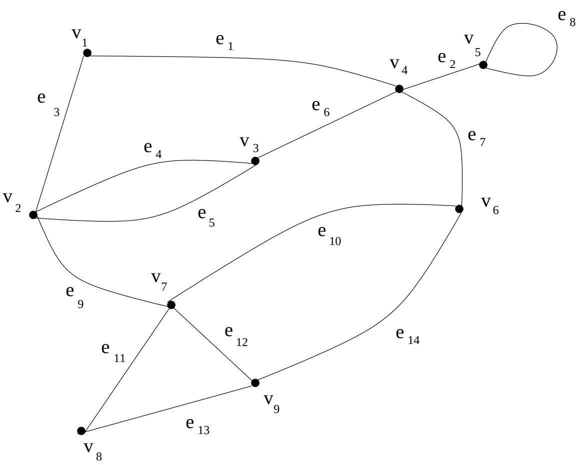
A *graph* $G = (V, E)$ is a pair of finite sets, a set V of *vertices* (singular *vertex*) and a set E of pairs of vertices called *edges*. We will typically write $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ where each $e_k = (v_i, v_j)$ for some v_i, v_j; in that case, v_i and v_j are called *end points* of the edge e_k. Graphs are drawn so that vertices are dots and edges are line segments or curves connecting two dots.

graph *vertex* *edge* *end points*

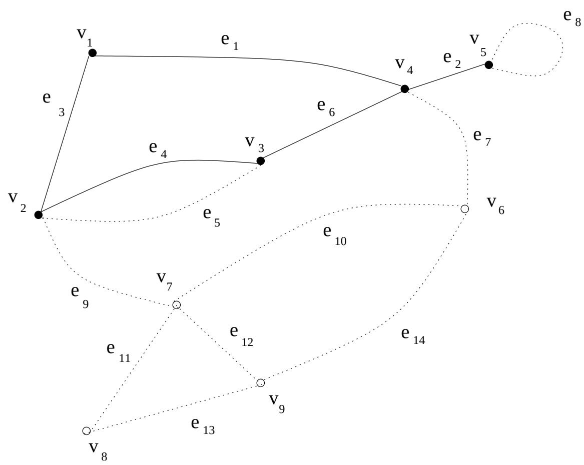
We call the edges *pairs* of vertices for lack of a better term; a pair is generally considered a two-tuple (in this case, it would be an element of $V \times V$); moreover, we write edges with parentheses and a comma, just as we would with tuples. However, we mean something slightly different. First, tuples are ordered. Second, an edge as a pair of vertices is not unique.

An edge (v_i, v_j) is *incident* on its end points v_i and v_j; we also say that it *connects* them. If vertices v_i and v_j are connected by an edge, they are *adjacent* to one another. If a vertex is adjacent to itself, that connecting edge is called a *self-loop*. If two edges connect the same two vertices, then those edges are *parallel* to each other. Below left, e_1 is incident on v_1 and v_4. e_{10} connects v_7 and v_6. v_9 and v_6 are adjacent. e_8 is a self-loop. e_4 and e_5 are parallel.

incident *connects* *adjacent* *self-loop* *parallel*



sample graph



subgraph that happens to be simple

The *degree* $\deg(v)$ of a vertex v is the number of edges incident on the vertex, with self-loops

degree

counted twice. $\deg(v_1) = 2$, $\deg(v_5) = 3$, and $\deg(v_2) = 4$. A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$ (and, by definition of graph, for any edge $(v_i, v_j) \in E'$, $v_i, v_j \in V'$). A graph $G = (V, E)$ is *simple* if it contains no parallel edges or self-loops. The graph $(\{v_1, v_2, v_3, v_4, v_5\}, \{e_1, e_2, e_3, e_4, e_6\})$, above right, is a simple subgraph of the graph shown.

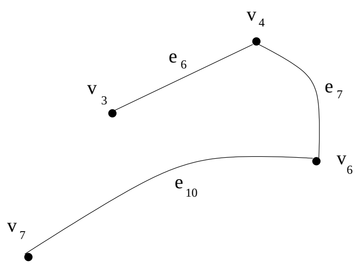
subgraph

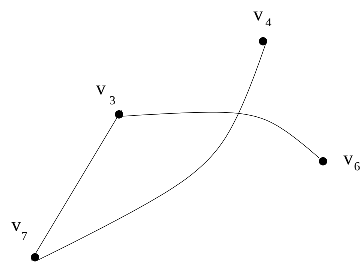
simple

A simple graph $G = (V, E)$ is *complete* if for all $v_i, v_j \in V$, the edge $(v_i, v_j) \in E$. The subgraph $(\{v_7, v_8, v_9\}, \{e_{11}, v_{12}, v_{13}\})$ is complete. The *complement* of a simple graph $G = (V, E)$ is a graph $\overline{G} = (V, E')$ where for $v_i, v_j \in V$, $(v_i, v_j) \in E'$ if $(v_i, v_j) \notin E$; in other words, the complement has all the same vertices and all (and only) those possible edges that are not in the original graph. The complement of the subgraph $(\{v_3, v_4, v_6, v_7\}, \{e_6, e_7, e_{10}\})$ is $(\{v_3, v_4, v_6, v_7\}, \{(v_3, v_7), (v_7, v_4), (v_3, v_6)\}$, as shown below. Recall that we have defined *complete* and *complement* only in terms of *simple* graphs, so self-loops are not considered.

complete

complement





A *walk* from vertex v to vertex w, $v, w \in V$, is a sequence alternating between vertices in V and edges in E, written $v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$ where $v_0 = v$ and $v_n = w$ and for all i, $1 \leq i < n$, $e_i = (v_{i-1}, v_i)$. (If a graph is simple, then it is possible to omit the edges when describing the path.) v is called the *initial* vertex and w is called the *terminal* vertex. A walk is *trivial* if it contains only one vertex and no edges; otherwise it is *nontrivial.* The *length* of a walk is the number of edges (not necessarily distinct, since an edge may appear more than once). In the graph below, some examples of non-trivial walks are $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ with length 6, $v_5e_{14}v_{15}$ with length 1, and $v_{11}e_{21}v_{12}e_{17}v_9e_{18}v_{13}e_{22}v_{12}e_{17}v_9e_{18}v_13e_{23}v_{14}$ with length 7.

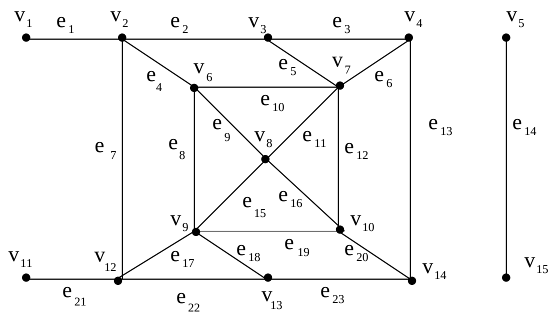
walk

initial

terminal

trivial

length



A graph is *connected* if for all $v, w \in V$, there exists a walk in G from v to w. This graph is not connected, since no walk exists from v_5 or v_{15} to any of the other vertices. However, the subgraph excluding v_5, v_{15}, and e_{14} is connected.

connected

A *path* is a walk that does not contain a repeated edge. $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ is a path, but $v_{11}e_{21}v_{12}e_{17}v_9e_{18}v_{13}e_{22}v_{12}e_{17}v_9e_{18}v_{13}$. is not. If the walk contains no repeated vertices, except possibly the initial and terminal, then the walk is *simple*. $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ is not simple, since v_6 occurs twice. Its subpath $v_8e_{11}v_7e_{10}v_6e_8v_9$ is simple.

path

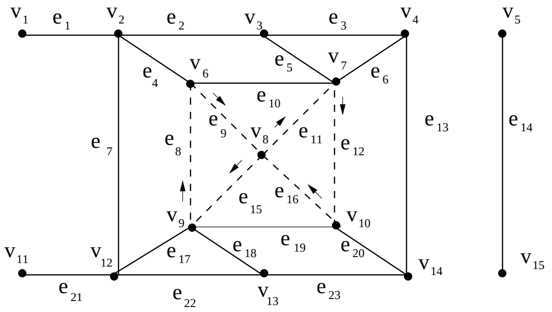
simple

If $v = w$ (that is, the initial and terminal vertices are the same), then the walk is *closed*. A *circuit* is a closed path. A *cycle* is a simple circuit. In the earlier example, $v_6e_9v_8e_{11}v_7e_{12}v_{10}e_{16}v_8e_{15}v_9e_8v_6$ is a circuit, but not a cycle, since v_8 is repeated. $v_2e_4v_6e_8v_9e_{17}v_{12}e_7v_2$ is a cycle.

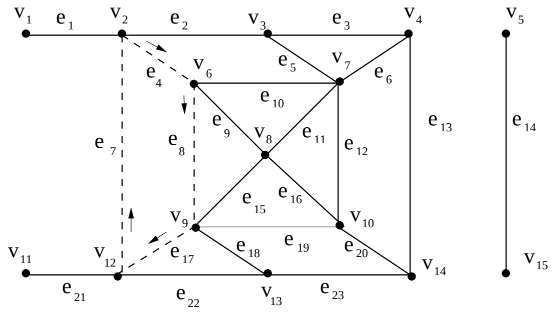
closed

circuit

cycle



Circuit

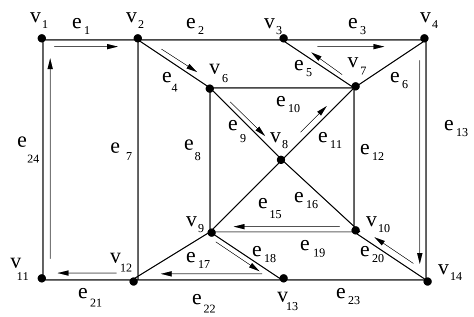


Cycle

An *Euler circuit* of G is a circuit that contains every vertex and every edge. (Since it is a circuit, this also means that an Euler circuit contains very edge exactly once. Vertices, however, may be repeated.) A *Hamiltonian cycle*, which for a graph $G = (V, E)$ is a cycle that includes every vertex in V. Since it is a cycle, this means that no vertex or edge is repeated; however, not all the edges need to be included. Here is a Hamiltonian cycle in a graph similar to the one at the beginning of this chapter (with the disconnected subgraph removed).

Euler circuit

Hamiltonian cycle



Proofs of graph theoretical propositions can get messy. They involve a lot of notation with edges and vertices. Paths can be annoying to reason about since they are written as sequences of subscripted v's and e's. To relieve some of the pain, we'll allow graph theory proofs to be a little less formal than the proofs in MATH/CSCI 243. Things like substitution and rules of arithmetic and algebra may be used uncited, for example. This should allow us to focus on the core of the proof. Consider this one:

Theorem 1 *If $G = (V, E)$ is a connected graph and for all $v \in V$, deg$(v) = 2$, then G is a cycle.*

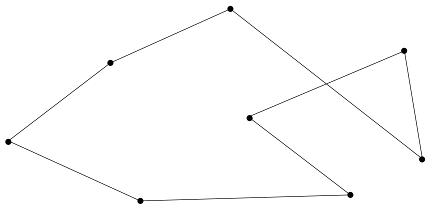
The important thing to think about is *what is the burden of this proposition?* In other words, what do we need to show? Identifying that will be an exercise in applying the definitions listed above, and it will give us a road map through the actual proof.

First of all, what we need to show is that G is a cycle. That means G *has* a cycle which happens to be all of G. This is our first step to unravelling what needs to be shown— it's a proof of existence. We must show *there exists* a cycle in G that comprises all of G.

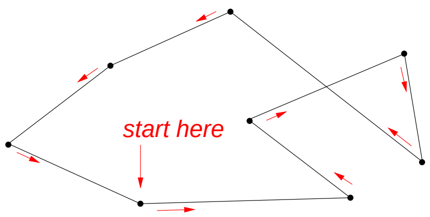
Now, what's a cycle? It's a simple circuit. *Simple* means it has no repeated internal vertices. What's a circuit? It's a closed path. *Closed* means it has the same initial and terminal vertex. A *path* is a walk with no repeated edges.

So, here's our proof outline or strategy: 1. Construct a walk. 2. Show that the walk has no repeated edges (so it's a path) 3. Show that it has the same first and last vertex (so it's closed—and it's also a circuit) 4. Show that it has no repeated internal vertices (so it's simple—and it's also a cycle) 5. Show that every vertex and edge in G is this cycle.

Now, why is this proposition true? Let's draw a picture of a connected graph, all of whose vertices are 2.



This theorem is almost obvious now. All we need to do is pick any vertex to begin with, and travel out by any edge. For ever vertex we get to, we just leave by the edge other than the one we entered.



Don't let this reasoning by example blind you to one special case:

Ready to prove?

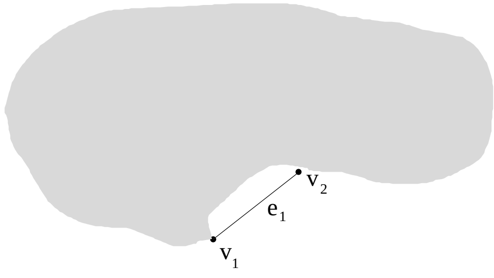
> **Proof.** Suppose $G = (V, E)$ is a connected graph and for all $v \in V$, $\deg(v) = 2$.
>
> First suppose $|V| = 1$, that is, there is only one vertex, v. Since $\deg(v) = 2$, this implies that there is only one edge, $e = (v, v)$. Then the cycle vev comprises the entire graph.

This looks like the beginning of a proof by induction, but actually it is a traditional division into cases. We are merely getting a special case out of the way. We want to use the fact that there can be no self-loops, but that is true only if there are more than one vertex.

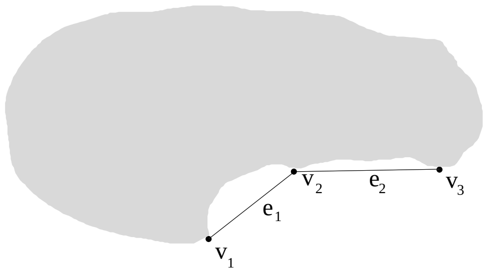
> Next suppose $|V| > 1$. By the exercise below, G has no self-loops.

We'll leave that part for you.

> Then construct a walk c in this manner: Pick a vertex $v_1 \in V$ and an edge $e_1 = (v_1, v_2)$. Since $\deg(v_1) = 2$, e must exist, and since G contains no self-loops, $v_1 \neq v_2$.



> Since $\deg(v_2) = 2$, there exists another edge, $e_2 = (v_2, v_3) \in E$.



Continue this process until we reach a vertex already visited, so that we can write $c = v_1 e_1 e_2 v_3 \ldots e_{x-1} v_x$ where $v_x = v_i$ for some i, $1 \leq i < x$. We will reach such a vertex eventually because V is finite.

Only one vertex in c is repeated, since reaching a vertex for the second time stops the building process. Hence c is simple.

Since we never repeat a vertex (until the last), each edge chosen leads to a new vertex, hence no edge is repeated in c, so c is a path.

We are always choosing the edge other than the one we took into a vertex, so $i \neq x - 1$.

Suppose $i \neq 1$. Since no other vertex is repeated, v_{i-1}, v_{i+1}, and v_{x-1} are distinct. Therefore, distinct edges (v_{i-1}, v_i), (v_i, v_{i+1}), and (v_{x-1}, v_i) all exist, and so $\deg(v_i) \geq 3$. Since $\deg(v_i) = 2$, this is a contradiction. Hence $i = 1$. Moreover, $v_1 = v_x$ and c is closed.

As a closed, simple path, c is a cycle.

Suppose that a vertex $v \in V$ is not in c, and let v' be any vertex in c. Since G is connected, there must be a walk, c' from v to v', and let edge e' be the first edge in c' (starting from v') that is not in c, and let v'' be an endpoint in c' in c. Since two edges incident on v'' occur in c, accounting for e' means that $\deg(v'') \geq 3$. Since $\deg(v_i) = 2$, this is a contradiction. Hence there is no vertex not in c.

Suppose that an edge $e \in E$ is not in c, and let v be an endpoint of e. Since v is in the cycle, there exist distinct edges e_1 and e_2 in c that are incident on v, implying $\deg(v) \geq 3$. Since $\deg(v) = 2$, this is a contradiction. Hence there is no edge not in c.

Therefore, c is a cycle that comprises the entire graph, and G is a cycle. □

Here's a summary of the terms:

simple (of a graph)	no self loops or parallel edges
walk	an alternating sequence of vertices and edges, starting and ending on a vertex
path	a walk without a repeated edge
simple (of a path)	no repeated vertex (except possibly the initial and terminal being the same)
closed	having the same vertex for the initial and terminal vertex
circuit	a closed path
cycle	a simple circuit
Euler circuit	a circuit containing every edge in the graph
Hamiltonian cycle	a cycle containing every vertex in the graph

Exercise. Show that if G is connected, for all $v \in V, \deg(v) = 2$, and $|V| > 1$, then G has no self-loops.