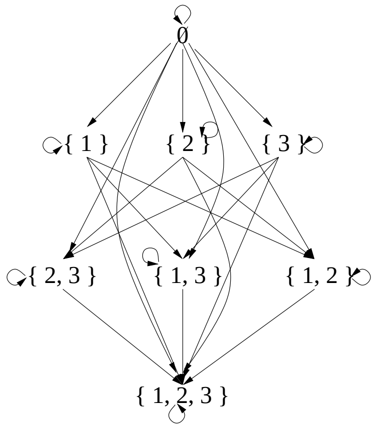
Lattices

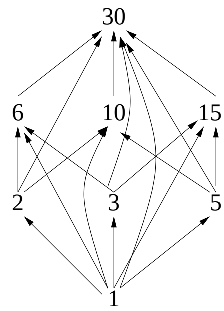
1 Review: Partial orders

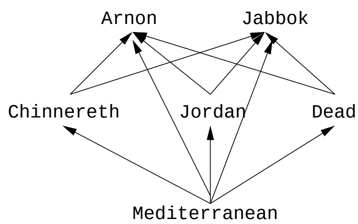
A *partial order relation* is a relation R on a set X that is reflexive, transitive, and antisymmetric. A set X on which a partial order is defined is called a *partially ordered set* or a *poset*. For examples, the set $\mathscr{P}(\{1,2,3\})$, that is, the powerset of $\{1,2,3\}$ or the set of all subsets of $\{1,2,3\}$ is a poset with relation $\subseteq$; and the set of positive divisors of 30 is a poset with relation $|$ (divides); the set of water bodies in and around biblical Israel is a poset with the relation `isWestOf`.

partial order relation

poset







The idea of the ordering being only partial is because not every pair of elements in the set is organized by it. In this case, for example, $\{1,2\}$ and $\{1,3\}$ are not comparable. We say that for a partial order relation $\preceq$ on a set A, $a, b \in A$ are *comparable* if $a \preceq b$ or $b \preceq a$. 6 and 30 are comparable for $|$ $(6|30)$, but 6 and 15 are not comparable. For $\subseteq$, $\emptyset$ is comparable to everything; in fact, it is a subset of everything. Arnon is not comparable to everything, but everything it is comparable with is west of it. These last observations lead us to say that if $\preceq$ is a partial order relation on A, then $a \in A$ is

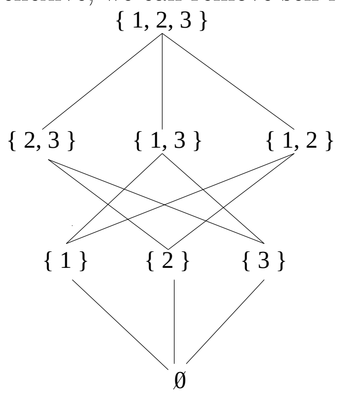
comparable

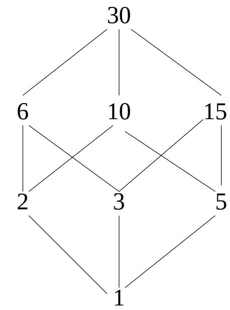
- *maximal* if $\forall\ b \in A$, $b \preceq a$ or b and a are not comparable. *maximal*
- *minimal* if $\forall\ b \in A$, $a \preceq b$ or b and a are not comparable. *minimal*

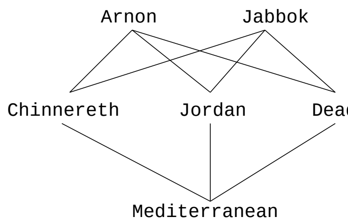
- *greatest* if $\forall\, b \in A$, $b \preceq a$. *greatest*
- *least* if $\forall\, b \in A$, $a \preceq b$ *least*

The graphs of partial orders become very cluttered. Instead, we use a pared down version of a graph called a *Hasse diagram*, after German mathematician Helmut Hasse. It strips out redundant information. To transform the graph of a partial order relation into a Hasse diagram, first draw it so that all the arrows (except for self-loops) are pointing up. Antisymmetry makes this possible. Then, since the arrangement on the page informs us what direction the arrows are going, the arrowheads themselves are redundant and can be erased. Finally, since we know that the relation is transitive and reflexive, we can remove self-loops and short-cuts.

Hasse diagram







2 Lattices defined

Suppose A is a poset with relation $\preceq$. Then a is an *upper bound* of a subset $B \subseteq A$ if for all $b \in B$, $b \preceq a$. Furthermore, a is an *least upper bound (LUB)* of a subset $B \subseteq A$ if for all upper bounds c of B, $a \preceq c$. Similarly we can define *lower bound* and *greatest lower bound (GLB)*.

A poset A with relation $\preceq$ is a *lattice* if for every pair of elements (or every subset with cardinality 2) has a LUB and and GLB. Suppose A is a lattice and $a, b \in A$. Then we define the *join* operator $\vee$ and the *meet* operator $\wedge$ to be the LUB and GLB, respectively, of a and b.

$\mathscr{P}(\{1, 2, 3\})$ with $\subseteq$ is a lattice; here $\vee = \cup$ and $\wedge = \cap$. The set of divisors of 30 with $|$ is a lattice; here $\vee$ is the least common multiple and $\wedge$ is the greatest common divisor. The set of water bodies of Israel is not a lattice since `Arnon` and `Jabbok` have no LUB.

If A with $\preceq$ is a lattice, then $S \in A$ is called a *sublattice* of A if for all $a, b \in S$, $a \vee b \in S$ and $a \wedge b \in S$. Notice that this basically means any subset of A that is itself a lattice. $\mathscr{P}(\{1, 2\}) = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$ is a sublattice of $\mathscr{P}(\{1, 2, 3\})$. The set $\{1, 3, 5, 15\}$, that is, the positive divisors of 15, is a sublattice of the set of positive divisors of 30.

upper bound

least upper bound (LUB)

lower bound

greatest lower bound (GLB)

lattice

sublattice

3 Lattice properties and results

Here are a variety of results about lattices, some proof sketches.

Theorem 1 *If A is a set, then $\mathscr{P}(A)$ with $\subseteq$ is a lattice.*

> **Proof sketch.** First, we need to show that it's a poset. It's pretty clear that $\subseteq$ is transitive, reflexive, and antisymmetric.
>
> Next suppose $B, C \in \mathscr{P}(A)$, that is, $B \subseteq A$ and $C \subseteq A$. Then $B \cup C$ is an upper bound for B and C (since $B \subseteq B \cup C$ and $C \subseteq B \cup C$). Now suppose $D \in \mathscr{P}(A)$ is an upper bound for C and B, in other words $B \subseteq D$ and $C \subseteq D$. Suppose $x \in B \cup C$. Then by definition of union, $x \in B$ or $x \in C$; either way, $x \in D$. Hence by definition of subset, $B \cup C \subseteq D$. Therefore $B \cup C$ is the LUB for B and C.

Similarly we can show that B and C have a GLB, and that shows that this is a lattice. □

Theorem 2 *Let D_n be the set of positive divisors of $n \in \mathbb{N}$. D_n is a lattice with $|$.*

Try to prove this yourself.

Theorem 3 *If A is a lattice with $\preceq$, then for all $a, b, c \in A$,*
a. $a \vee b = b$ *iff* $a \preceq b$
b. $a \wedge b = a$ *iff* $a \preceq b$
c. $a \wedge b = a$ *iff* $a \vee b =\preceq b$

Proof. (a) ($\Rightarrow$) Suppose $a \vee b = b$. That means b is the LUB of b. By definition of LUB, $a \preceq b$.

$(Leftarrow)$ Suppose $a \preceq b$. Since $\preceq$ is reflexive, $b \preceq b$, so b is an upper bound for a and b. Now suppose c is an upper bound for a and b, that is $a \preceq c$ and $b \preceq c$. The fact that $b \preceq c$ is enough to show that b is the LUB. Hence $a \vee b = b$.

Try (b) and (c) on your own. □

Theorem 4 *If A is a lattice, then the operators $\vee$ and $\wedge$ are idempotent (ie, $a \vee a = a$), commutative, and associative.*

Proof avoidance. Pretty trivial in the definitions of $\vee$ and $\wedge$. □

A lattice A is *distributive* if for all $a, b, c \in A$, *distributive*

$$a \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c) a \wedge (b \vee c) = (a \vee b) \wedge (a \vee c)$$

Theorem 5 *If A is a set, the lattice $\mathscr{P}(A)$ with $\subseteq$ is distributive.*

Proof. $\cup$ and $\cap$ are distributive, see Exercise 4 of Chapter 11 (2006: pg 81; 2007: pg 91). □