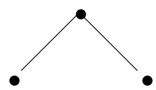
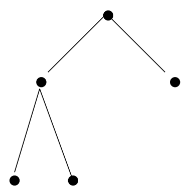
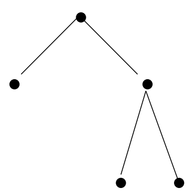
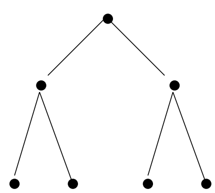
5.4 Structural induction

Observe the following about full binary trees:

Tree	Nodes	Links
	1	0
	3	2
	5	4
	5	4
	7	6

Clearly every full binary tree has one more node than it has links. An intuitive explanation is not too hard, either; we can pair up each node with the link above it, and this will account for every link and also every node, except for the root. Thus there is one more node than there are links. But how can we prove it formally?

Another way to state our observation takes into account that trees are built from smaller trees. Any time we make a new tree, we take two other trees and link them together with a new root. This adds two new links and one new node. If the two older trees already each had one more node than links, the new tree will also.

invariant

This property is an *invariant*, a proposition that does not vary through changing circumstances. In this case, the holding of the property for the tree depends on the property holding on subtrees. We have a new format of proof, this for an invariant over a recursively-defined mathematical structure.

Theorem 5.1 *For any full binary tree T,* $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

> **Proof.** Suppose T is a full binary tree.

This is all we are given. We need to use the definition to analyze what this means.

> By definition of full binary tree, T is either a single node or a node with links to two full binary trees.

Two possibilities. This calls for division into cases. We will use special names for these cases, based on how they correspond to cases in the recursive definition of full binary tree.

Base case. Suppose T is a single node. Then it has one node (itself) and no links, that is $\texttt{nodes}(T) = 1$ and $\texttt{links}(T) = 0$. Thus $\texttt{nodes}(T) = \texttt{links}(T) + 1$.

Inductive case. Suppose T is a node with links to two other full binary trees, call them T_1 and T_2. Since T adds one node and two links to the subtrees, $\texttt{nodes}(T) = \texttt{nodes}(T_1) + \texttt{nodes}(T_2) + 1$ and $\texttt{links}(T) = \texttt{links}(T_1) + \texttt{links}(T_2) + 2$.

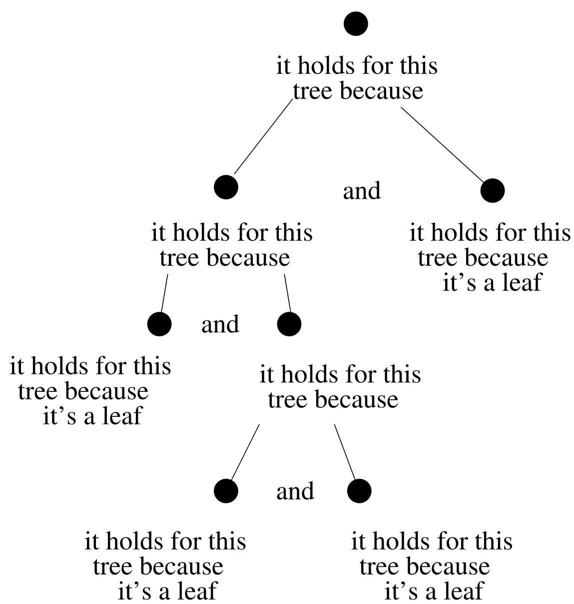
Here is the new part. From what we said earlier, we know T_1 and T_2 each have one more node than links. How do we know that, formally? The theorem itself tells us.

By structural induction, $\texttt{nodes}(T_1) = \texttt{links}(T_1) + 1$ and $\texttt{nodes}(T_2) = \texttt{links}(T_2) + 1$. Then

$$\begin{array}{lcll} \texttt{nodes}(T) & = & \texttt{nodes}(T_1) + \texttt{nodes}(T_2) + 1 & \text{as stated above} \\ & = & \texttt{links}(T_1) + 1 + \texttt{links}(T_2) + 1 + 1 & \text{by substitution} \\ & = & \texttt{links}(T_1) + \texttt{links}(T_2) + 2 + 1 & \text{by algebra} \\ & = & \texttt{links}(T) + 1 & \text{by substitution} \end{array}$$

Either way, $\texttt{nodes}(T) = \texttt{links}(T) + 1$. □

Why does that work? It is the same principle behind recursive algorithms and recursive structures. We can apply the proof of this theorem to the subtrees, which requires it to be applied to their subtrees, until we reach the leaves.



In our proof, we cite this as "by *structural induction*." This is the proof technique which performs a division into cases based on the structure of a recursively defined set. The proposition we proved can be broken down to a predicate *structural induction*

$$I(T) = \texttt{nodes}(T) = \texttt{links}(T) + 1$$

wrapped in a universally quantified proposition

$$\forall\, T \in \mathcal{T}, I(T).$$

where $\mathcal{T}$ is the set of full binary trees. Therefore, propositions which are predicates universally quantified over a recursively-defined set are candidates for proof by structural induction.

Exercises

Prove using structural induction.

5.4.1 For any full binary tree T, $\texttt{leaves}(T) = \texttt{internals}(T) + 1$.

5.4.2 For any fully binary tree T, $\texttt{leaves}(T) \leq 2^{\texttt{height}(T)}$.

5.4.3 For any full binary tree T, $\texttt{nodes}(T) \leq 2^{\texttt{height}(T)+1} - 1$.

5.4.4 For any full binary tree T, $\texttt{nodes}(T)$ is odd. (Prove this directly, using structural induction. Do not use Theorem 5.1 or an earlier exercise as a lemma.)

5.4.5 Let the set S be defined so that for all $s \in S$, either

- $s = 3$, or
- $s = t + u$ for some $t, u \in S$.

Then for all $s \in S$, $3|s$.

5.5 Mathematical induction

This section is one of our occasional forays into the world of integers. Consider numbers in the form $3^n - 1$ for $n \in \mathbb{W}$.

n	0	1	2	3	4	5	6	7	8
$3^n - 1$	0	2	8	26	80	242	728	2186	6560

From these examples, one might expect that all numbers in that form are even. Not a surprising result, in fact, since by eyeballing it we can tell that 3^n will be odd and so $3^n - 1$ will be even. Let us wrap that in a predicate

$$I(n) = 3^n - 1 \text{ is even.}$$

And our claim becomes $\forall\ n \in \mathbb{W}, I(n)$.

If we take a specific example, say $3^4 - 1$, we see

$$3^4 - 1 = 3 \cdot 3^3 - 1 = 3 \cdot (3^3 - 1 + 1) - 1 = 3 \cdot (3^3 - 1) + 3 - 1 = 3 \cdot (3^3 - 1) + 2$$

This might seem like a haphazard rearranging of values, but what it does is relate $3^4 - 1$ to $3^3 - 1$, and moreover relates $I(4)$ to $I(3)$. If we knew $I(3)$ were true, a simple manipulation would yield $I(4)$.

Lemma 5.1 *If $I(3)$, then $I(4)$.*

Proof. Suppose $I(3)$, that is, $3^3 - 1$ is even. By definition of even, $3^3 - 1 = 2 \cdot k$ for some $k \in \mathbb{Z}$. Then,

$$\begin{array}{rcll} 3^4 - 1 & = & 3 \cdot (3^3 - 1) + 2 & \text{as we showed above} \\ & = & 3 \cdot (2 \cdot k) + 2 & \text{by substitution} \\ & = & 2 \cdot (3 \cdot k + 1) & \text{by algebra} \end{array}$$

Since $3 \cdot k + 1 \in \mathbb{Z}$, $3^4 - 1$ is even by definition. $\square$

In Section 5.1, we learned a recursive construction for the whole numbers. In Section 5.4, we learned a proof technique for propositions universally quantified over recursively-defined sets. Let us put these things together to prove

Theorem 5.2 *For all $n \in \mathbb{W}$, $3^n - 1$ is even.*

Proof. Suppose $n \in \mathbb{W}$. By definition of whole number (from Section 5.1), either $n = 0$ or $n = m + 1$ where $m \in \mathbb{W}$.

Base case: Suppose $n = 0$. Then $3^n - 1 = 3^0 - 1 = 1 - 1 = 0 = 2 \cdot 0$, which is even by definition.