11.8 Boolean algebras

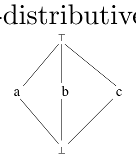
Recall that if A is a bounded lattice and $a \in A$, then the *complement* of a is an element $a' \in A$ such that *complement*

$$a \vee a' = \top \qquad \text{and} \qquad a \wedge a' = \bot$$

This definition, of course, does not imply that such a complement *exists*, nor that it is unique if it exists. We did, however, show the following theorem.

Theorem 11.7 *If A is a bounded, distributive lattice and $a \in A$, then a', if it exists, is unique.*

Now, we define a bounded lattice A to be *complemented* if for all $a \in A$, a' exists. The following lattice is complemented, since a, b, and c are complements of each other, and $\top$ and $\bot$ are complements of each other. (Notice that this is one of our prototypical non-distributive lattices) *complemented*



We have this result.

Theorem 11.8 *Complementedness is an isomorphic invariant.*

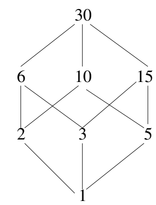
Proof. Suppose $A, \preceq$ and $B, \preceq'$ are isomorphic lattices with isomorphism $f : A \leftarrow B$. Suppose further that A is complemented.

Suppose $b \in B$, and let $a = f^{-1}(b)$. Since A is complemented, there exists $a' \in A$ such that $a \vee a' = \top_A$ and $a \wedge a' = \bot_A$. Then

$$\begin{aligned}
b \vee f(a') &= f(f^{-1}(b) \vee a') && \text{by your Exercise 3} \\
&= f(a \vee a') && \text{by substitution} \\
&= f(\top_A) && \text{by substitution} \\
&= \top_B && \text{since } \top \text{ and } \bot \text{ match across isomorphisms}
\end{aligned}$$

The fact that $b \wedge f(a') = \bot_B$ is similar. Hence $f(a')$ is a complement of b, and B is complemented. $\square$

A *boolean algebra* is a a complemented, distributive lattice. (Notice that being complemented implies being bounded. All finite lattices are bounded, but not all bounded lattices are finite. Nevertheless, we will assume *finite* boolean algebras for our discussion.) Both our running examples, D_{30} and $\mathscr{P}(2,3,5)$, are boolean algebras. Take D_{30}. 30 and 1 are complements; 6 and 5 are complements; 15 and 2 are complements; 10 and 3 are complements (notice a pattern?). *boolean algebra*



Corollary 11.1 *If A is a boolean algebra, then for all $a \in A$, a has a unique complement.*

> **Proof.** Since boolean algebras are complemented, at least one complement, a', exists. Since boolean algebras are distributive, Theorem 11.7 tells us that a' is unique. □

$\mathbb{B}$ Let $\mathbb{B}$ be the set of boolean values, { true, false }. Let $\mathbb{B}_n$ be the set of n-tuples over $\mathbb{B}$. For example, $\mathbb{B}_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$. (We call "false" 0 and "true" 1 and omit the commas and parentheses from tuple notation in order to reduce clutter. "000" would conventionally be written "(false, false, false)". We will also sometimes treat the values as the numbers 0 and 1 rather than the boolean values true and false.) If $a \in \mathbb{B}_n$, then we refer to the components of a as $a_1, \ldots, a_n$.

Define $\preceq$ so that for $a, b \in \mathbb{B}_n$, $a \preceq b$ if for all i, $1 \leq i \leq n$, $a_i \leq b_i$ (interpreting the components as numbers). It's straightforward to see that $\mathbb{B}_n, \preceq$ is a lattice.