## 8.3 Strolling about a graph

The obvious way to imagine or visualize a graph is to think of the vertices as places and the edges as connections or bridges between those places. Accordingly, we can imagine ourselves or an object moving around in a graph from vertex to vertex.

In fact, that seems to have been the original way graph theory was though of. Many people consider graph theory to have been begun by a problem proposed by mathematician Leonhard Euler about crossing bridges while walking around town. More on that problem later.

*walk*

A *walk* from vertex $v$ to vertex $w$, $v, w \in V$, is a sequence alternating between vertices in $V$ and edges in $E$, written $v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$ where $v_0 = v$ and $v_n = w$ and for all $i$, $1 \leq i < n$, $e_i = (v_{i-1}, v_i)$. You could also think of a walk simply as a sequence of edges, but it is more difficult to describe the constraint that way; we certainly could describe the path by listing only the edges. (If a graph is simple, then it is possible to omit the edges when describing the path.) $v$ is called the *initial* vertex and $w$ is called the *terminal* vertex.

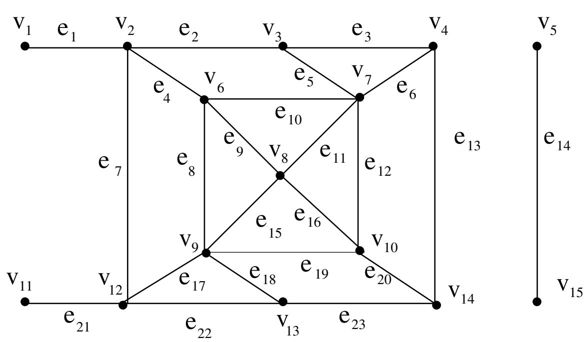
*initial*

*terminal*

*trivial*

*length*

A walk is *trivial* if it contains only one vertex and no edges; otherwise it is *nontrivial*. The *length* of a walk is the number of edges (not necessarily distinct, since an edge may appear more than once). In the graph below, some examples of non-trivial walks are $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ with length 6, $v_5e_{14}v_{15}$ with length 1, and $v_{11}e_{21}v_{12}e_{17}v_9e_{18}v_{13}e_{22}v_{12}e_{17}v_9e_{18}v_13e_{23}v_{14}$ with length 7.




*connected*

A graph is *connected* if for all $v, w \in V$, there exists a walk in $G$ from $v$ to $w$. (Notice that for any vertex there exists a trivial walk to itself, so we do not need self loops for a graph to be connected.) This graph is not connected, since no walk exists from $v_5$ or $v_{15}$ to any of the other vertices. However, the subgraph excluding $v_5$, $v_{15}$, and $e_{14}$ is connected.

*path*

A *path* is a walk that does not contain a repeated edge. $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ is a path, but $v_{11}e_{21}v_{12}e_{17}v_9e_{18}v_{13}e_{22}v_{12}e_{17}v_9e_{18}v_{13}$. is not. If the walk contains no repeated vertices, except possibly the initial and terminal, then the walk is *simple*. $v_1e_1v_2e_4v_6e_9v_8e_{11}v_7e_{10}v_6e_8v_9$ is not simple, since $v_6$ occurs twice. Its subpath $v_8e_{11}v_7e_{10}v_6e_8v_9$ is simple.

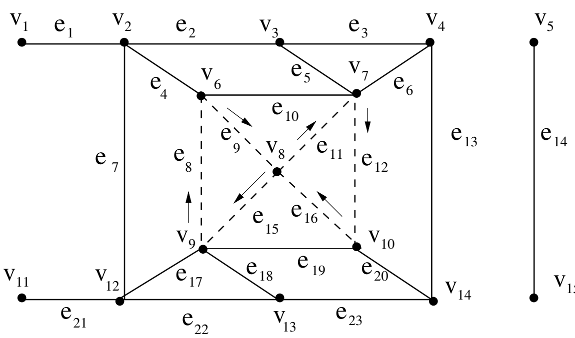
*simple*

If $v = w$ (that is, the initial and terminal vertices are the same), then the walk is *closed*. A *circuit* is a closed path. A *cycle* is a simple circuit. In the earlier example, $v_6e_9v_8e_{11}v_7e_{12}v_{10}e_{16}v_8e_{15}v_9e_8v_6$ is a circuit, but not a cycle, since $v_8$ is repeated. $v_2e_4v_6e_8v_9e_{17}v_{12}e_7v_2$ is a cycle.

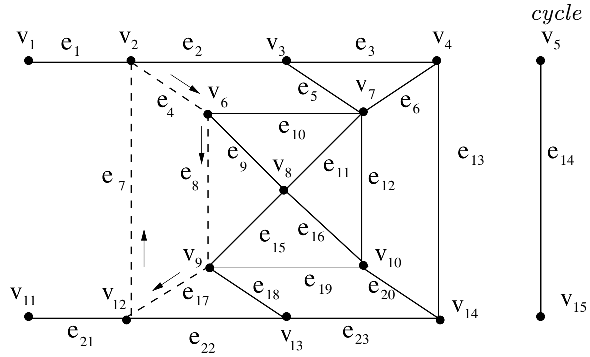
*closed*

*circuit*

*cycle*




Circuit




Cycle

Proofs of graph theoretical propositions can get messy. They involve a lot of notation with edges and vertices. Paths can be annoying to reason about since they are written as sequences of subscripted $v$'s and $e$'s. To relieve some of the pain, we'll allow graph theory proofs to be a little less formal. Things like substitution and rules of arithmetic and algebra may be used uncited, for example. This should allow us to focus on the core of the proof. Consider this one:

**Theorem 8.3** *If $G = (V, E)$ is a connected graph and for all $v \in V$, $deg(v) = 2$, then $G$ is a cycle.*

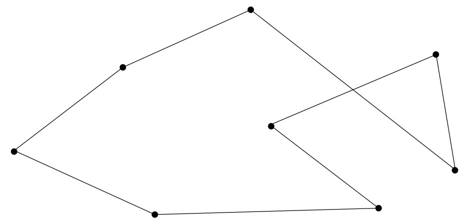
The important thing to think about is *what is the burden of this proposition?* In other words, what do we need to show? Identifying that will be an exercise in applying the definitions listed above, and it will give us a road map through the actual proof.

First of all, what we need to show is that $G$ is a cycle. That means $G$ *has* a cycle which happens to be all of $G$. This is our first step to unravelling what needs to be shown— it's a proof of existence. We must show *there exists* a cycle in $G$ that comprises all of $G$.

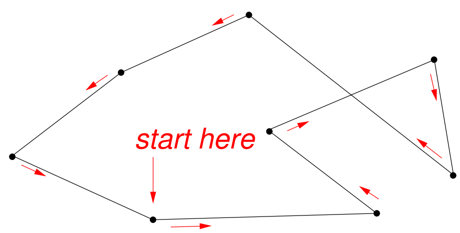
Now, what's a cycle? It's a simple circuit. *Simple* means it has no repeated internal vertices. What's a circuit? It's a closed path. *Closed* means it has the same initial and terminal vertex. A *path* is a walk with no repeated edges.

So, here's our proof outline or strategy: 1. Construct a walk. 2. Show that the walk has no repeated edges (so it's a path) 3. Show that it has the same first and last vertex (so it's closed—and it's also a circuit) 4. Show that it has no repeated internal vertices (so it's simple—and it's also a cycle) 5. Show that every vertex and edge in $G$ is this cycle.

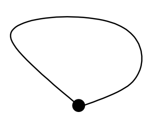
Now, why is this proposition true? Let's draw a picture of a connected graph, all of whose vertices are 2.



This theorem is almost obvious now. All we need to do is pick any vertex to begin with, and travel out by any edge. For ever vertex we get to, we just leave by the edge other than the one we entered.




Don't let this reasoning by example blind you to one special case:



Ready to prove?

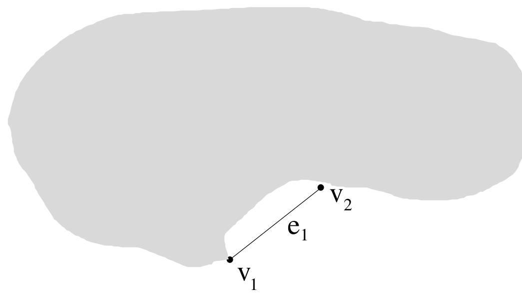
> **Proof.** Suppose $G = (V, E)$ is a connected graph and for all $v \in V$, $\deg(v) = 2$. First suppose $|V| = 1$, that is, there is only one vertex, $v$. Since $\deg(v) = 2$, this implies that there is only one edge, $e = (v, v)$. Then the cycle $vev$ comprises the entire graph.

This looks like the beginning of a proof by induction, but actually it is a traditional division into cases. We are merely getting a special case out of the way. We want to use the fact that there can be no self-loops, but that is true only if there are more than one vertex.

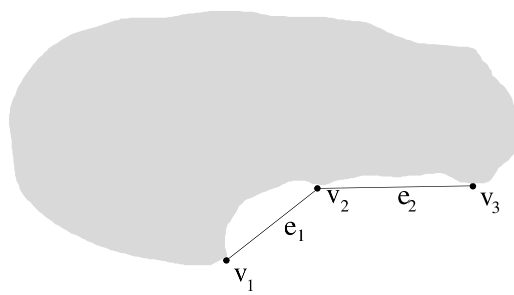
> Next suppose $|V| > 1$. By the exercise below, $G$ has no self-loops.

We'll leave that part for you.

> Then construct a walk $c$ in this manner: Pick a vertex $v_1 \in V$ and an edge $e_1 = (v_1, v_2)$. Since $\deg(v_1) = 2$, $e$ must exist, and since $G$ contains no self-loops, $v_1 \neq v_2$.




Since $\deg(v_2) = 2$, there exists another edge, $e_2 = (v_2, v_3) \in E$.




Continue this process until we reach a vertex already visited, so that we can write $c = v_1 e_1 e_2 v_3 \ldots e_{x-1} v_x$ where $v_x = v_i$ for some $i$, $1 \leq i < x$. We will reach such a vertex eventually because $V$ is finite.

Only one vertex in $c$ is repeated, since reaching a vertex for the second time stops the building process. Hence $c$ is simple.

Since we never repeat a vertex (until the last), each edge chosen leads to a new vertex, hence no edge is repeated in $c$, so $c$ is a path.

We are always choosing the edge other than the one we took into a vertex, so $i \neq x - 1$.

Suppose $i \neq 1$. Since no other vertex is repeated, $v_{i-1}$, $v_{i+1}$, and $v_{x-1}$ are distinct. Therefore, distinct edges $(v_{i-1}, v_i)$, $(v_i, v_{i+1})$, and $(v_{x-1}, v_i)$ all exist, and so $\deg(v_i) \geq 3$. Since $\deg(v_i) = 2$, this is a contradiction. Hence $i = 1$. Moreover, $v_1 = v_x$ and $c$ is closed.

As a closed, simple path, $c$ is a cycle.

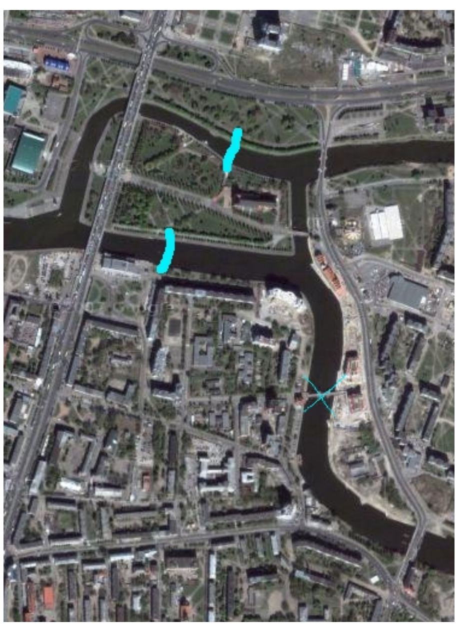
Suppose that a vertex $v \in V$ is not in $c$, and let $v'$ be any vertex in $c$. Since $G$ is connected, there must be a walk, $c'$ from $v$ to $v'$, and let edge $e'$ be the first edge in $c'$ (starting from $v'$) that is not in $c$, and let $v''$

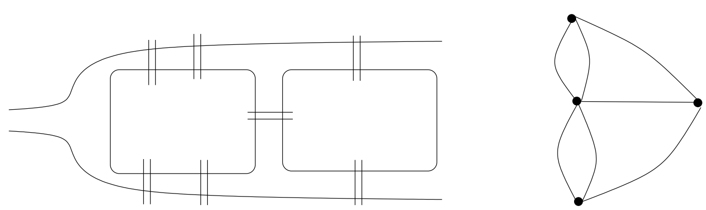
be an endpoint in $c'$ in $c$. Since two edges incident on $v''$ occur in $c$, accounting for $e'$ means that $\deg(v'') \geq 3$. Since $\deg(v_i) = 2$, this is a contradiction. Hence there is no vertex not in $c$.

Suppose that an edge $e \in E$ is not in $c$, and let $v$ be an endpoint of $e$. Since $v$ is in the cycle, there exist distinct edges $e_1$ and $e_2$ in $c$ that are incident on $v$, implying $\deg(v) \geq 3$. Since $\deg(v) = 2$, this is a contradiction. Hence there is no edge not in $c$.

Therefore, $c$ is a cycle that comprises the entire graph, and $G$ is a cycle. □

Leonhard Euler proposed a problem based on the bridges in the town of Königsberg, Prussia (now Kaliningrad, Russia). Two branches in of the Pregel River converge in the town, delineating it into a north part, a south part, an east part, and an island in the middle, as shown below. In Euler's time, the east part had one bridge to each the north part and the south part, the island had two bridges each to the north part and the south part, and one bridge connected the east part and the island. Supposing your house was in any of the four parts, is it possible to walk around town (beginning and ending at your house) and cross every bridge exactly once?





To the left is an arial view from Google maps of modern Kaliningrad showing some of the bridges still in approximately the same spot as in the eighteenth century, but the locations of two older, no longer extant, bridges indicated and a newer bridge exed out. Above is a more abstract map and a graph representation.

We can turn this into a graph problem by representing the information with a graph whose vertices stand for the parts of town and whose edges stand for the bridges, as displayed above. Let $G = (V, E)$ be a graph. An *Euler circuit* of $G$ is a circuit that contains every vertex and every edge. (Since it is a circuit, this also

*Euler circuit*

means that an Euler circuit contains very edge exactly once. Vertices, however, may be repeated.) The question now is whether or not this graph has an Euler circuit. We can prove that it does not, and so such a stroll about town is impossible.

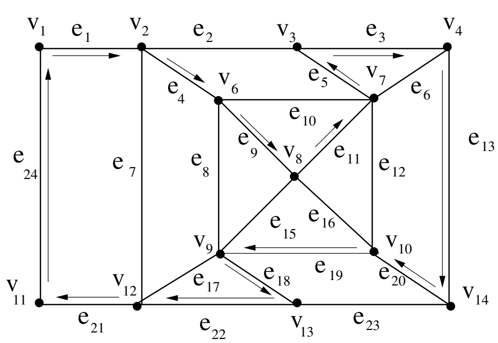
**Theorem 8.4** *If a graph $G = (V, E)$ has an Euler circuit, then every vertex of $G$ has an even degree.*

**Proof.** (To be done in class.)

The northern, eastern, and southern parts of town each have odd degrees, so by the contrapositive of this theorem, no Euler circuit around town exists.

*Hamiltonian cycle*

Another interesting case is that of a *Hamiltonian cycle*, which for a graph $G = (V, E)$ is a cycle that includes every vertex in $V$. Since it is a cycle, this means that no vertex or edge is repeated; however, not all the edges need to be included. Here is a Hamiltonian cycle in a graph similar to the one at the beginning of this chapter (with the disconnected subgraph removed).




Here is a summary of the terms in this section, first adjectives that apply to walk and nouns that are various kinds of walks.

| **Adjectives** | |
|---|---|
| Trivial | Having only one vertex and no edges. |
| Simple | Having no repleated vertices (except, possibly, the initial and terminal). |
| Closed | Having the same vertex as initial and terminal. |

| Nouns | |
|---|---|
| Walk | An alternating sequence of vertices and edges, each edge coming between its end points. |
| Path | A walk with no repeated *edge* (repeated vertices are ok). |
| Subpath | A path that is part of another path. |
| Circuit | A closed path (no repeated edges, initial and terminal the same) |
| Cycle | A simple circuit (no repeated edges or vertices, except the initial and terminal, which are the same) |
| Euler circuit | A circuit containing every vertex and every edge (repeated vertices are still ok). |
| Hamiltonian cycle | A cycle containing every vertex (still no repeated edges or vertices; missing edges are ok). |

**Exercises**

8.3.1 Draw a graph that has a simple walk that is not a path.

Prove. Assume $G = (V, E)$ is a graph.

8.3.2 If $G$ is connected, then $|E| \geq |V| - 1$.

8.3.3 If $G$ is connected, $|V| \geq 2$, and $|V| > |E|$, then $G$ has a vertex of degree 1.

8.3.4 If $G$ is not connected, then $\overline{G}$ is connected.

8.3.5 If $G$ is connected, for all $v \in V$, $\deg(v) = 2$, and $|V| > 1$, then $G$ has no self-loops.

8.3.6 Every circuit in $G$ contains a subwalk that is a cycle.

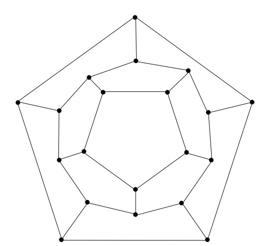
8.3.7 If for all $v \in V$, $\deg(v) \geq 2$, then $G$ contains a cycle.

8.3.8 If $v, w \in V$ are part of a circuit $c$ and $G'$ is a subgraph of $G$ formed by removing one edge of $c$, then there exists a path from $v$ to $w$ in $G'$.

8.3.9 If $G$ has no cycles, then it has a vertex of degree 0 or 1.

8.3.10 If $G$ has an Euler circuit, then it is connected.

8.3.11 Find a Hamiltonian cycle in the following graph (copy it and highlight the walk on your copy).



8.3.12 If $G$ has a non-trivial Hamiltonian cycle, then $G$ has a subgraph $G' = (V', E')$ such that

- $G'$ contains every vertex of $G$ ($V' = V$)
- $G'$ is connected,
- $G'$ has the same number of edges as vertices ($|V'| = |E'|$), and
- every vertex of $G'$ has degree 2.

8.3.13 Reinterpret the Königsburg bridges problem by making a graph whose vertices represent the bridges and whose edges represent land connections between bridges. What problem are you solving now?