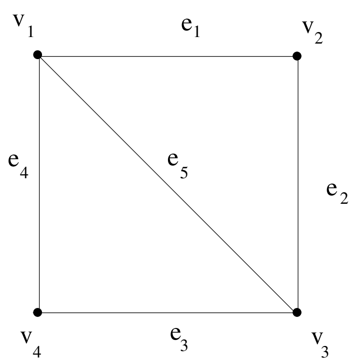
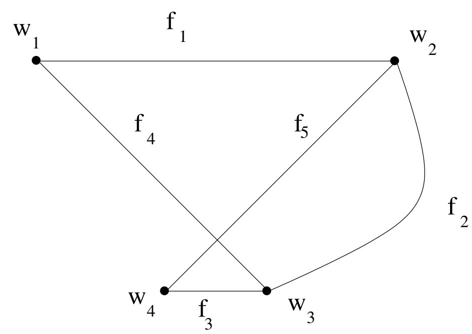
8.4 Isomorphisms

We have already seen that the printed shape of the graph—the placement of the dots, the resulting angles of the lines, any curvature of the lines—is not of the essence of the graph. The only things that count are the names of the vertices and edges and the abstract shape, that is, the connections that the edges define. However, consider the two graph representations below, which illustrate the graphs $G = (V = \{v_1, v_2, v_3, v_4\}, E = \{e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_4), e_4 = (v_4, v_1), e_5 = (v_1, v_3)\})$ and $G' = (W = \{w_1, w_2, w_3, w_4, w_5\}, F = \{f_1 = (w_1, w_2), f_2 = (w_2, w_3), f_3 = (w_3, w_4), f_4 = (w_3, w_1), f_5 = (w_4, w_2)\})$.





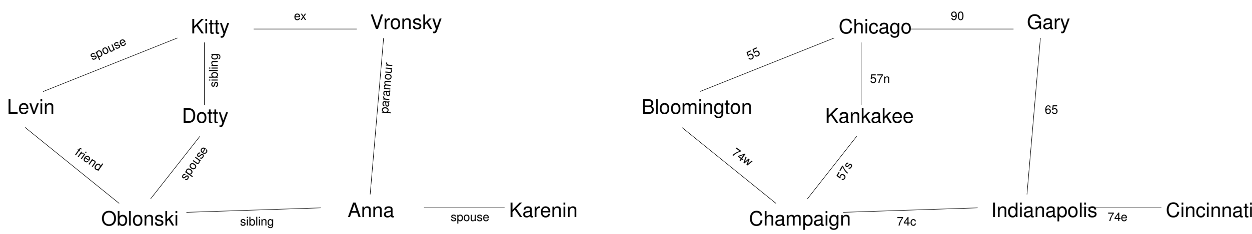
These graphs have much in common. Both have four vertices and five edges. Both have two vertices with degree two and two vertices with degree three. Both have a Hamiltonian cycle ($v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$ and $w_1 f_1 w_2 f_5 w_4 f_3 w_3 f_4 w_1$, leaving out e_5 and f_2, respectively) and two other cycles (involving e_5 and f_2). In fact, if you imagine switching the positions of w_1 and w_2, with the edges sticking to the vertices as they move, and then doing a little stretching and squeezing, you could transform the second graph until it appears identical to the first.

In other words, these two really are the same graph, in a certain sense of sameness. The only difference is the arbitrary matter of names for the vertices and edges. We can formalize this by writing renaming functions, $g : V \to W$ and $h : E \to F$.

v	$g(v)$
v_1	w_2
v_2	w_4
v_3	w_3
v_4	w_1

e	$h(e)$
e_1	f_5
e_2	f_3
e_3	f_4
e_4	f_1
e_5	f_2

There is another way of looking at this. The practical use of graphs is that they model information. These two graphs model the dummy information contained in the labels of the vertices and edges, and those labels are different so in that sense the graphs are different. However, they have the same *structure*. To revise an earlier example, compare the graph of the characters in *Anna Karenina* with a map of the connections among a few midwestern cities.



Again, the structure of these two graphs is the same. The difference is the meaning. Structure and meaning make an important distinction is many fields of study; for example, in linguistics, syntax (or grammar) deals with the structure of language, while semantics considers meaning.

isomorphism

The term for this kind of equivalence between graphs is *isomorphism*, from the Greek roots *iso* meaning "same" and *morphë* meaning "shape." This is a way to recognize identical abstract shapes of graphs, that two graphs are the same up to renaming.

What would we need to do to formulate a formal definition of *isomorphism*? Note that if two graphs are isomorphic, we can talk about that as a relation, "is isomorphic to." What properties do you suspect that relation has?