Ex 5.3.7. Prove that if R is a relation on a set A and $(a, b) \in R$, then $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$.

Proof. *Suppose R is a relation over A and that $(a, b) \in R$.*

[Note that $(a, b) \in R$ implies that both a and b must be elements of A.]

Suppose $x \in \mathcal{I}_R(b)$. By definition of image, $(b, x) \in R$. Since $(a, b) \in R$, we have $(a, x) \in R \circ R$ by definition of composition. Moreover $x \in \mathcal{I}_{R \circ R}(a)$ by definition of image.

Therefore $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$ by definition of subset. □

Ex 5.3.9. Prove that if R is a relation from A to B, then $i_B \circ R = R$.

Proof. *First suppose* $(x, y) \in i_B \circ R$. *By definition of composition, there exists* $b \in B$ *such that* $(x, b) \in R$ *and* $(b, y) \in i_B$.

By definition of the identity relation, $b = y$. *By substitution,* $(x, y) \in R$. *Hence* $i_B \circ R \subseteq R$ *by definition of subset.*

Next suppose $(x, y) \in R$. *By how* R *is defined, we know* $x \in A$ *and* $y \in B$.

By definition of the identity relation, $(y, y) \in i_B$. *By definition of composition,* $(x, y) \in i_B \circ R$. *Hence* $R \subseteq i_B \circ R$.

Therefore, by definition of set equality, $i_B \circ R = R$. $\square$