Theorem *$\mathscr{H}_{pm}$ is universal.*

Proof. *Suppose p and m as specified earlier. Suppose $k, \ell \in$ Keys, and $h_{ab} \in \mathscr{H}_{pm}$ (which implies supposing that $a \in [1, p)$ and $b \in [0, p)$).*
Let $r = (a \cdot k + b) \mod p$ and $s = (a \cdot \ell + b) \mod p$
Subtracting gives us

$$\begin{aligned} r - s &\equiv (a \cdot k + b) - (a \cdot \ell + b) && \mod p \\ &\equiv a \cdot (k - \ell) && \mod p \end{aligned}$$

Now a cannot be 0 because $a \in [1, p)$. Similarly $k - \ell$ cannot be 0, since $k \neq \ell$. Hence $a \cdot (k - \ell) \neq 0$.
Since p is prime and greater than a, k, and ℓ, it cannot be a factor of $a \cdot (k - \ell)$. In other words, $a \cdot (k - \ell) \mod p \neq 0$. By substitution, $r - s \neq 0$, and so $r \neq s$.
By another substitution, $(a \cdot k + b) \mod p \neq (a \cdot \ell + b) \mod p$.

Define the following function, given k and ℓ, which maps from (a, b) pairs to (r, s) pairs (formally, $[1, p) \times [0, p) \to [1, p) \times [0, p)$):

$$\phi_{k\ell}(a, b) = ((a \cdot k + b) \mod p, (a \cdot \ell + b) \mod p)$$

Now consider the inverse of that function.

$$\begin{aligned} \phi_{k\ell}^{-1}(r, s) &= (((r - s) \cdot (k - \ell)^{-1}) \mod p), (r - ak) \mod p) \\ &= (a, b) \end{aligned}$$

The existence of ϕ^{-1} implies that ϕ is a one-to-one correspondence. Hence for each (a, b) pair, there is a unique (r, s) pair. Since the pair (a, b) specifies a hash function, that means that for each hash function in the family $\mathscr{H}_{pm}$, there is a unique (r, s) pair.

There are $p-1$ possible choices for a and p choices for b, so there are $p \cdot (p-1)$ hash functions in family $\mathscr{H}_{pm}$. Likewise there are p choices for r, and for each r there are $p-1$ choices for s (since $s \neq r$). Thus we can partition the set $\mathscr{H}_{pm}$ into p subsets by r value, each subset having $p-1$ hash functions.

For a given r, at most one out of every m can have an s that is equivalent to r mod m, in other words, at most $\frac{p-1}{m}$ hash functions.

Now sum that for all p of the subsets of $\mathscr{H}_{pm}$, and we find that the number of hash functions for which k and ℓ collide are

$$p \cdot \frac{p-1}{m} = \frac{p \cdot (p-1)}{m} = \frac{|\mathscr{H}_{pm}|}{m}$$

Therefore $\mathscr{H}_{pm}$ is universal by definition. $\square$

Theorem [Probability of any collisions.] *If Keys is a set of keys, $m = |Keys|^2$, p is a prime greater than all keys, and $h \in \mathscr{H}_{pm}$, then the probability that any two distinct keys collide in h is less than $\frac{1}{2}$.*

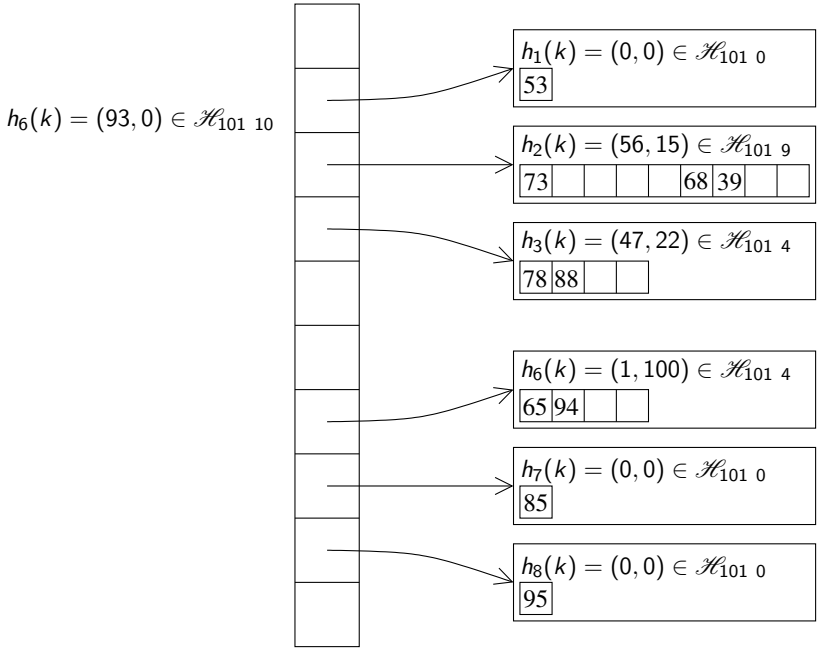
Proof. *Suppose we have a set Keys, $m = |Keys|^2$, p is a prime greater than all keys, and $h \in \mathscr{H}_{pm}$.*
Consider the number of pairs of unique keys. The number of pairs of keys is

$$\binom{n}{2} = \frac{n!}{2! \cdot (n-2)!} = \frac{n!}{2 \cdot (n-2)!} = \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{2 \cdot \cancel{(n-2)!}} = \frac{n \cdot (n-1)}{2}$$

Since $\mathscr{H}_{pm}$ is universal, each pair collides with probability $\frac{1}{m}$. Multiply that by the number of pairs, and the expected number of collisions is

$$\begin{aligned} \frac{n\cdot(n-1)}{2}\cdot\frac{1}{m} &< \frac{n^2}{2}\cdot\frac{1}{m} && \text{since } n\cdot(n-1) < n^2 \\ &= \frac{n^2}{2}\cdot\frac{1}{n^2} && \text{since } m = n^2 \\ &= \frac{1}{2} && \text{by cancelling } n^2 \end{aligned}$$

With the expected number of collisions less than one half, the probability there are any collisions is also less than $\frac{1}{2}$. $\square$


$h_6(k) = (93, 0) \in \mathscr{H}_{101\ 10}$
$h_1(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
53
$h_2(k) = (56, 15) \in \mathscr{H}_{101\ 9}$
73
68
39
$h_3(k) = (47, 22) \in \mathscr{H}_{101\ 4}$
78
88
$h_6(k) = (1, 100) \in \mathscr{H}_{101\ 4}$
65
94
$h_7(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
85
$h_8(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
95