







Lemma (Safe edges in Kruskal's algorithm.)

If $G = (V, E)$ is a graph, A is a subset of a minimum spanning tree for G , (u, v) is the lightest edge connecting any distinct connected components of A , then (u, v) is a safe edge for A , that is, $A \cup \{(u, v)\}$ is a subset of a minimum spanning tree.

Proof. Suppose everything in the hypothesis, in particular that A is a subset of some minimum spanning tree T and that u and v are in distinct connected components of A , call them A_u and A_v . Let w_T be the total weight of T , that is, the sum of the weights of all the edges of T . We want to prove that adding (u, v) to A makes something that is still a subset of some minimum spanning tree.

If $(u, v) \in T$, then we're done. Suppose, then, that T does not contain (u, v) . Since T is a spanning tree, it means that u and v are connected in T . Pick the lightest edge on the path from u to v that is not in A , call it (x, y) . Essentially (x, y) is an edge that was picked instead of (u, v) that contributed to connecting A_u and A_v .

Snip out (x, y) . This would disconnect T , that is, the graph $T - \{(x, y)\}$ is not a tree, but rather contains two connected components, one with u in it and the other with v in it. Now splice in (u, v) . That will reconnect u and v and make it into a tree again. Formally we've made a new spanning tree $(T - \{(x, y)\}) \cup \{(u, v)\}$.

The hypothesis says that (u, v) was the lightest edge connecting distinct components of A . That means $w(u, v) \leq w(x, y)$. That in turn means that the total weight of the new spanning tree is also just as good, if not better, than the old one: $w((T - \{(x, y)\}) \cup \{(u, v)\}) \leq w_T$. Since it ties or beats a (supposed) minimum spanning tree, $(T - \{(x, y)\}) \cup \{(u, v)\}$ must be a minimum spanning tree. Therefore (u, v) is safe. \square

		Kruskal		Prim	
		Unoptimized	Optimized	Unoptimized	Optimized
Sparse	Adjacency list	31579	28841	72364	58089
	Adjacency matrix	49128	35493	67887	49537
Medium	Adjacency list	147527	54877	180407	113555
	Adjacency matrix	127485	59821	146358	75906
Dense	Adjacency list	136762	69867	191617	123762
	Adjacency matrix	162468	78154	130984	72245