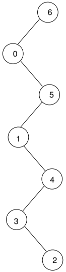
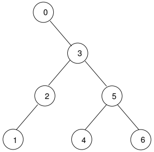
6, 0, 5, 1, 4, 2, 3



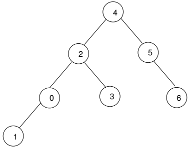
height 7
total depth 21
ANI 4

0, 3, 5, 2, 6, 1, 4



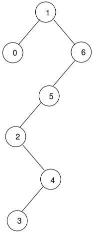
height 4
total depth 14
ANI 3

4, 2, 5, 3, 0, 1, 6



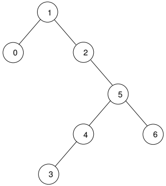
height 4
total depth 11
ANI 2.57

1, 6, 5, 2, 4, 3, 0



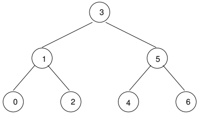
height 6
total depth 16
ANI 3.29

1, 2, 5, 4, 3, 0, 6

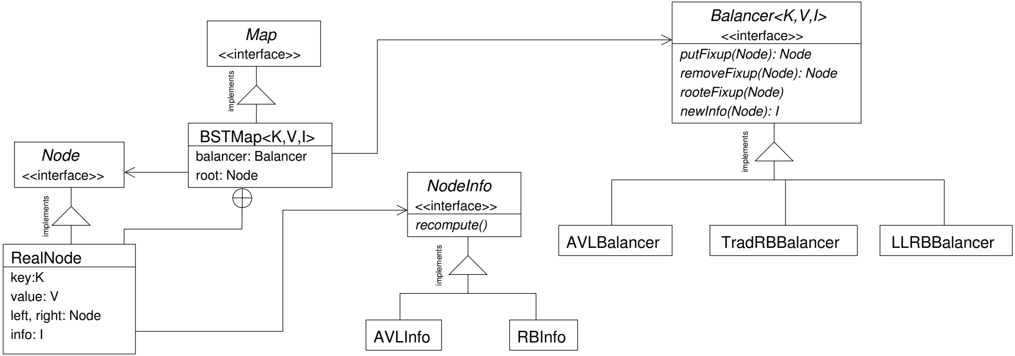


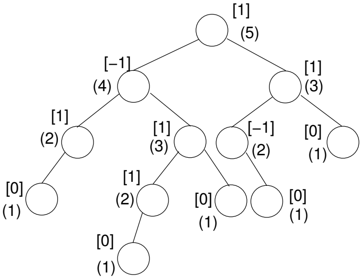
height 5
total depth 14
ANI 3

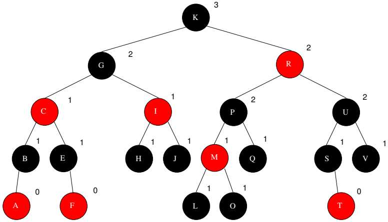
3, 1, 5, 0, 2, 4, 6

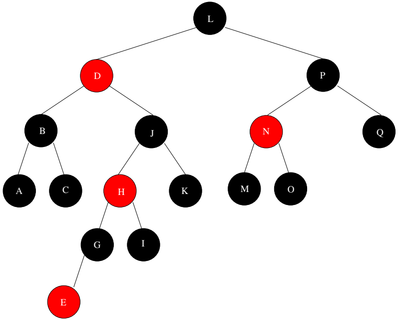


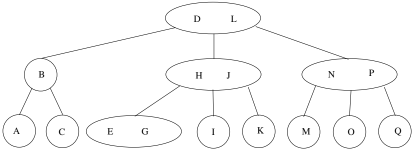
height 3
total depth 10
ANI 2.43

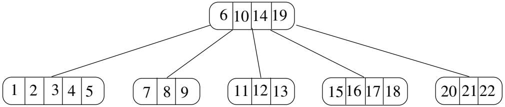

Map
<<interface>>
implements
BSTMap<K,V,I>
balancer: Balancer
root: Node
Node
<<interface>>
implements
RealNode
key:K
value: V
left, right: Node
info: I
NodeInfo
<<interface>>
recompute()
implements
AVLInfo
RBInfo
Balancer<K,V,I>
<<interface>>
putFixup(Node): Node
removeFixup(Node): Node
rooteFixup(Node)
newInfo(Node): I
implements
AVLBalancer
TradRBBalancer
LLRBBalancer

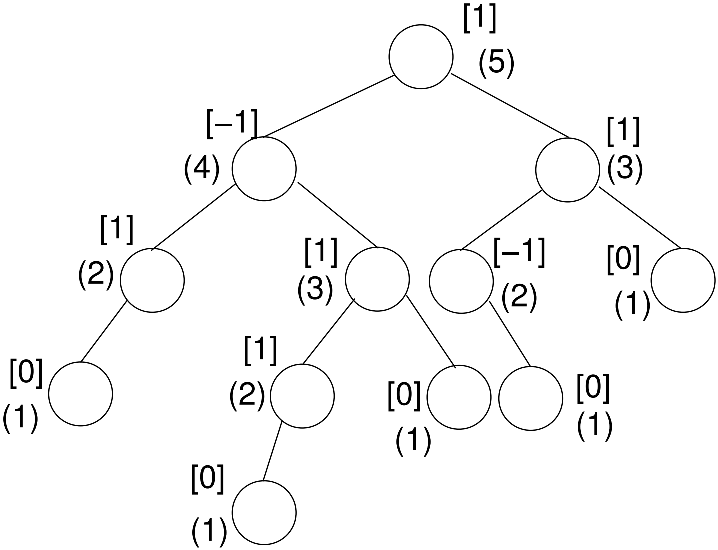

[1]
(5)
[−1]
(4)
[1]
(3)
[1]
(2)
[1]
(3)
[−1]
(2)
[0]
(1)
[0]
(1)
[1]
(2)
[0]
(1)
[0]
(1)
[0]
(1)

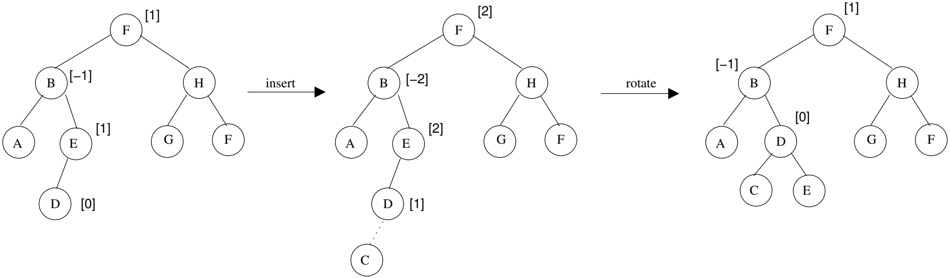

K 3
G 2
R 2
C 1
I 1
P 2
U 2
B 1
E 1
H 1
J 1
M 1
Q 1
S 1
V 1
A 0
F 0
L 1
O 1
T 0

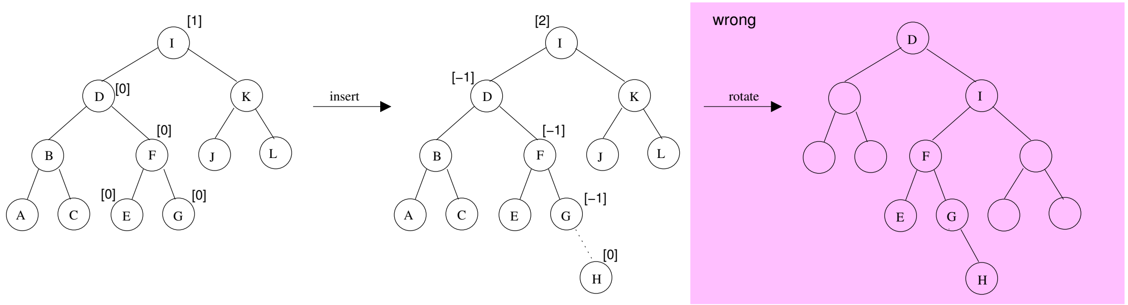

L
D
P
B
J
N
Q
A
C
H
K
M
O
G
I
E

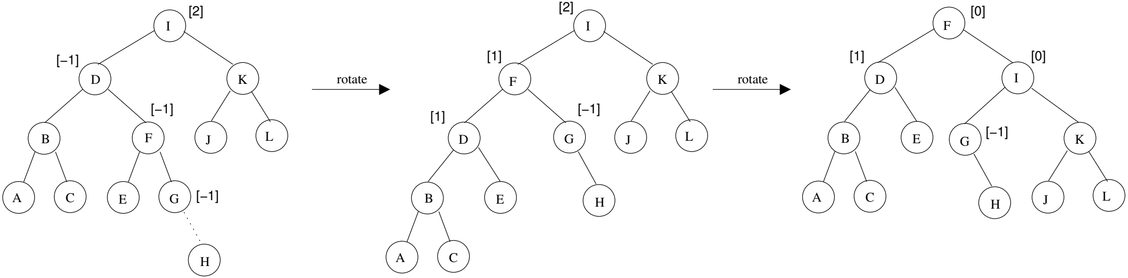

D L
B
H J
N P
A
C
E G
I
K
M
O
Q


6 10 14 19
1 2 3 4 5
7 8 9
11 12 13
15 16 17 18
20 21 22


[1]
(5)
[–1]
(4)
[1]
(3)
[1]
(2)
[1]
(3)
[–1]
(2)
[0]
(1)
[0]
(1)
[1]
(2)
[0]
(1)
[0]
(1)
[0]
(1)

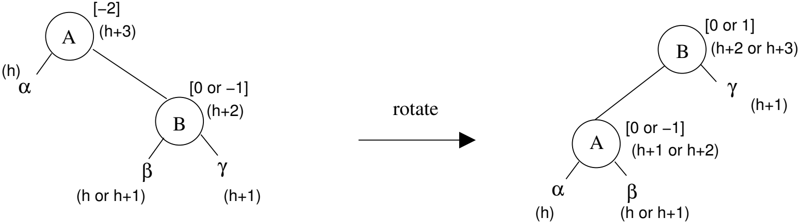

F [1]
B [–1]
H
A
E [1]
G
F
D [0]
insert
F [2]
B [–2]
H
A
E [2]
G
F
D [1]
C
rotate
F [1]
[–1] B
H
A
[0] D
G
F
C
E


[1]
I
[0]
D
K
[0]
B
F
J
L
[0]
A
C
E
G
[0]
insert
[2]
I
[–1]
D
K
[–1]
B
F
J
L
[–1]
A
C
E
G
[0]
H
wrong
rotate
D
I
F
E
G
H

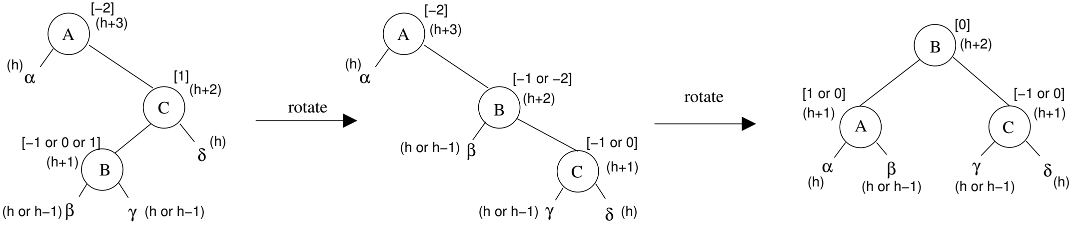


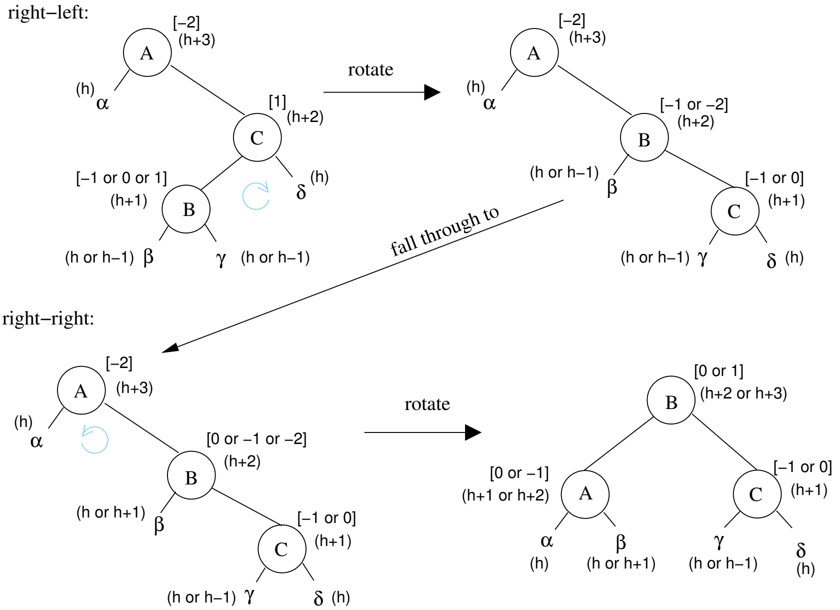
[2]
I
[–1]
D
K
[–1]
B
F
J
L
A
C
E
G
[–1]
H
rotate
[2]
I
[1]
F
K
[1]
D
[–1]
G
J
L
B
E
H
A
C
rotate
[0]
F
[1]
D
[0]
I
B
E
G
[–1]
K
A
C
H
J
L

right−right alone:



right–left alone:




right–left:
[–2]
(h+3)
A
(h)
α
C
[1]
(h+2)
[–1 or 0 or 1]
(h+1)
B
δ
(h)
(h or h–1)
β
γ
(h or h–1)
rotate
[–2]
(h+3)
A
(h)
α
[–1 or –2]
(h+2)
B
(h or h–1)
β
[–1 or 0]
(h+1)
C
(h or h–1)
γ
δ
(h)
fall through to
right–right:
[–2]
(h+3)
A
(h)
α
[0 or –1 or –2]
(h+2)
B
(h or h+1)
β
[–1 or 0]
(h+1)
C
(h or h–1)
γ
δ
(h)
rotate
[0 or 1]
(h+2 or h+3)
B
[0 or –1]
(h+1 or h+2)
A
[–1 or 0]
(h+1)
C
α
(h)
β
(h or h+1)
γ
(h or h–1)
δ
(h)

Invariant 25 (Postconditions of `RealNode.put()` with `AVLBalancer`.)
Let x be the root of a subtree on which `put()` is called and y be the node returned, that is, the root of the resulting subtree. The subtree rooted at y has no violations and the height of the subtree rooted at y is equal to or one greater than the original height of the subtree rooted at x.

Proof. *Suppose `put()` is called on node x in a BST using AVL balancing which has no violations. There are three cases: x is `nully`, x is a `RealNode` containing the key being searched for, or x is a `RealNode` with a different key. We use structural induction with the first two cases as base cases.*

Base case 1. *Suppose x is `nully`, which has height 0 Then the node y returned is a new `RealNode` with `nully` as both children, height 1, and balance 0. The subtree rooted at y has no violations and height one greater than the original height of x.*

Base case 2. *Suppose x is a `RealNode` whose key is equal to the key used for this `put()`. Then the value at node x is overwritten but node x itself is returned (so $y = x$ in this case) with the tree structure unchanged.*

Inductive case. *Suppose x is a `RealNode` and, without loss of generality, the key used for this `put()` is greater than the key at x, and so `put()` is called on the right child of x. Let h_0 be the height of the tree rooted at x before this call to `put()` on the right child, and let z the root of the subtree that results from this call to `put()` on the right child. Our inductive hypothesis is that the subtree rooted at z has no violations and that its height is equal to or one greater than the height of the original right child of x.*

Let h_1 be the height of the tree rooted at x after the call to `put()` on the right child but before the call to `putFixup()` with x.

Since since at most the height of its right subtree has increased by one, either $h_1 = h_0$ or $h_1 = h_0 + 1$. By supposition, the balance of x before the call to `put()` was no less than -1, since we supposed the tree had no violations. Since at most the height of its right subtree has increased by one, the balance of x is now no less than -2. We now have two subcases: Either the balance of x is greater than -2 or it is equal to -2.

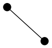
Suppose the balance of x is greater than -2. Then the call to `putFixup()` with x returns x unchanged, which is also returned as the result of `put()` (again $y = x$), a tree with no violations and height h_1.

On the other hand, suppose the balance of x is equal to -2. Then y is a node other than x returned by `putFixup()`. Let h_2 be the height of the subtree rooted at y when `putFixup()` returns. By inspection of the right-right and right-left subcases given above, the subtree rooted at y has no violations and either $h_2 = h_1$ or $h_2 = h_1 - 1$. In either of those cases $h_2 = h_0$ or $h_2 = h_0 + 1$.

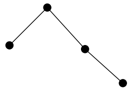
$\square$

A_1

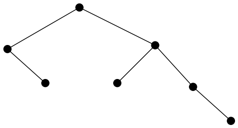
A_2



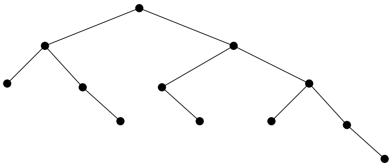
A_3



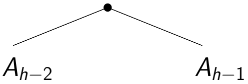
A_4



A_5



A_h



$$B_h = \begin{cases} 1 & \text{if } h = 1 \\ 2 & \text{if } h = 2 \\ B_{h-2} + B_{h-1} + 1 & \text{otherwise} \end{cases} \qquad B_h{+}1 = \begin{cases} 2 & \text{if } h = 1 \\ 3 & \text{if } h = 2 \\ (B_{h-2} + 1) + (B_{h-1} + 1) & \text{otherwise} \end{cases}$$

h	1	2	3	4	5	6
$B_h + 1$	2	3	5	8	13	21
B_h	1	2	4	7	12	20

$$
\begin{aligned}
B_h + 1 &> \frac{\phi^{h+2}}{\sqrt{5}} - 1 \\
B_h + 2 &> \frac{\phi^{h+2}}{\sqrt{5}} \\
\sqrt{5}(B_h + 2) &> \phi^{h+2} \\
h + 2 &< \log_\phi(\sqrt{5}B_h) \\
h &< \log_\phi(\sqrt{5}B_h) - 2 \\
&= \log_\phi B_h + \log_\phi \sqrt{5} - 2 \\
&= \frac{1}{\lg \phi} \lg B_h + \log_\phi \sqrt{5} - 2
\end{aligned}
$$