Chapter 7 outline:

- Introduction, function equality, and anonymous functions (last week Monday)
- Image and inverse images (last week Wednesday)
- Function properties, composition, and applications to programming (last week Friday)
- Cardinality (Monday)
- *Practice quiz* and Countability (**Today**)
- Review (Monday, Apr 18)
- Test 3, on Ch 6 & 7 (Wednesday, Apr 20)

Two finite sets X and Y have the *the same cardinality* as each other if there exists a one-to-one correspondence from X to Y.

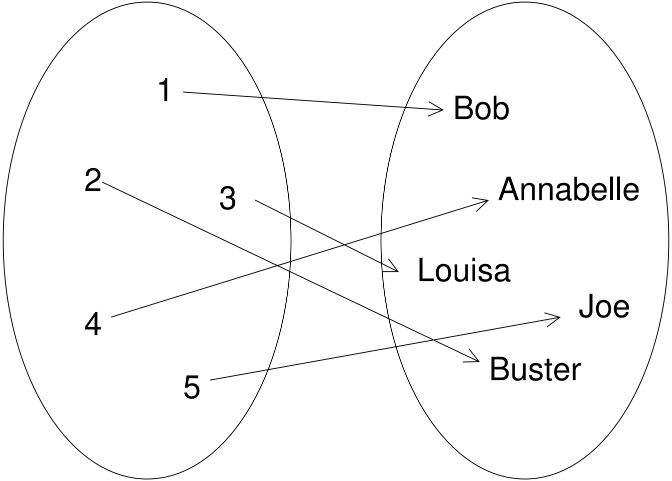
To use this *analytically*:

Suppose X and Y have the same cardinality. Then let f be a one-to-one correspondence from X to Y.
f is both onto and one-to-one.

To use this *synthetically*:

Given sets X and Y ...
[Define f] Let $f : X \to Y$ be a function defined as ...
Suppose $y \in Y$. *Somehow find $x \in X$ such that $f(x) = y$.* Hence f is onto.
Suppose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. *Somehow show $x_1 = x_2$.* Hence f is one-to-one.
Since f is a one-to-one correspondence, X and Y have the same cardinality as each other.

A finite set X has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \ldots n\}$ to X. Moreover, $|\emptyset| = 0$.



Two finite sets X and Y have the *the same cardinality* as each other if there exists a one-to-one correspondence from X to Y.

A finite set X has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \dots n\}$ to X. Moreover, $|\emptyset| = 0$.

Given a set X, if there exists $n \in \mathbb{N}$ and a one-to-one correspondence from $\{1, 2, \dots n\}$ to X, then X is *finite* and $|X| = n$. Otherwise, X is *infinite*.

Are all infinities equal?

Which is more intuitive to you,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{R}|$$

or

$$|\mathbb{N}| < |\mathbb{W}| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|$$

?

Thm 7.19. $\mathbb{W}$ and $\mathbb{N}$ have the same cardinality.

Proof. *[We need a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{W}$.]*
Let $f : \mathbb{W} \to \mathbb{N}$ be defined so that $f(n) = n + 1$.
Suppose $n \in \mathbb{N}$. Then $f(n-1) = (n-1) + 1 = n$, so f is onto.
Next suppose $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$, and moreover $n_1 = n_2$. Hence f is one-to-one.
Since a one-to-one correspondence exists between $\mathbb{W}$ and $\mathbb{N}$, the two sets have the same cardinality. □

A set X is *countably infinite* if there exists a one-to-one correspondence from $\mathbb{N}$ to X.

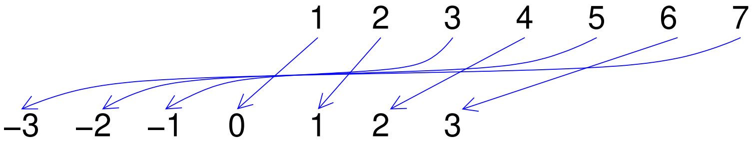
A set is *countable* if it is finite or countably infinite. Otherwise it is *uncountable*.

Thm 7.20. $\mathbb{Z}$ is countably infinite.

Proof. *[We need a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{Z}$.]*

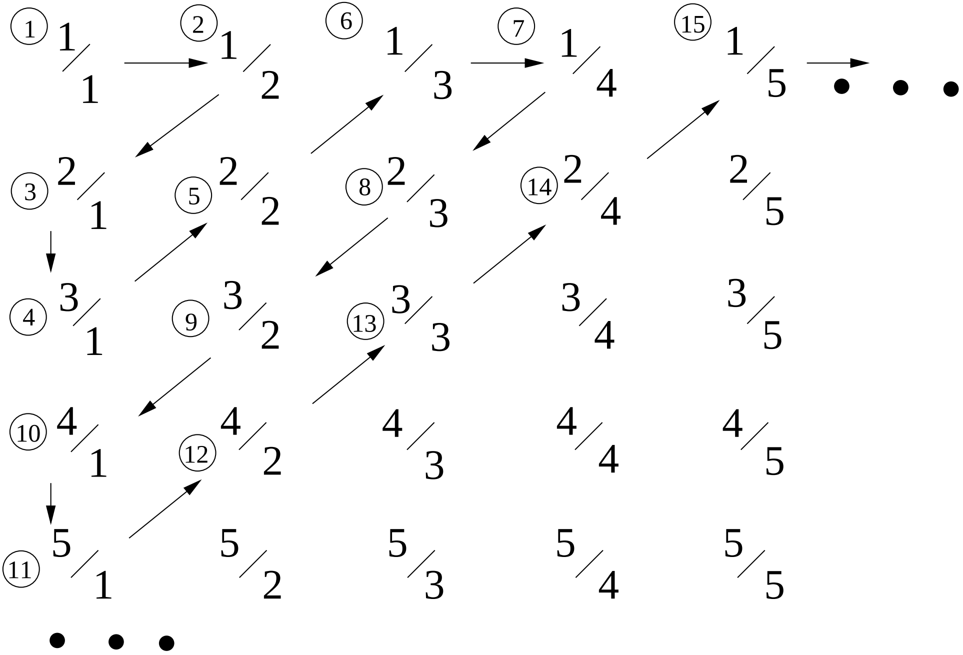
Let $f : \mathbb{N} \to \mathbb{Z}$ be defined so that

$$f(x) = \begin{cases} n \texttt{ div } 2 & \text{if } n \text{ is even} \\ -(n \texttt{ div } 2) & \text{otherwise} \end{cases}$$



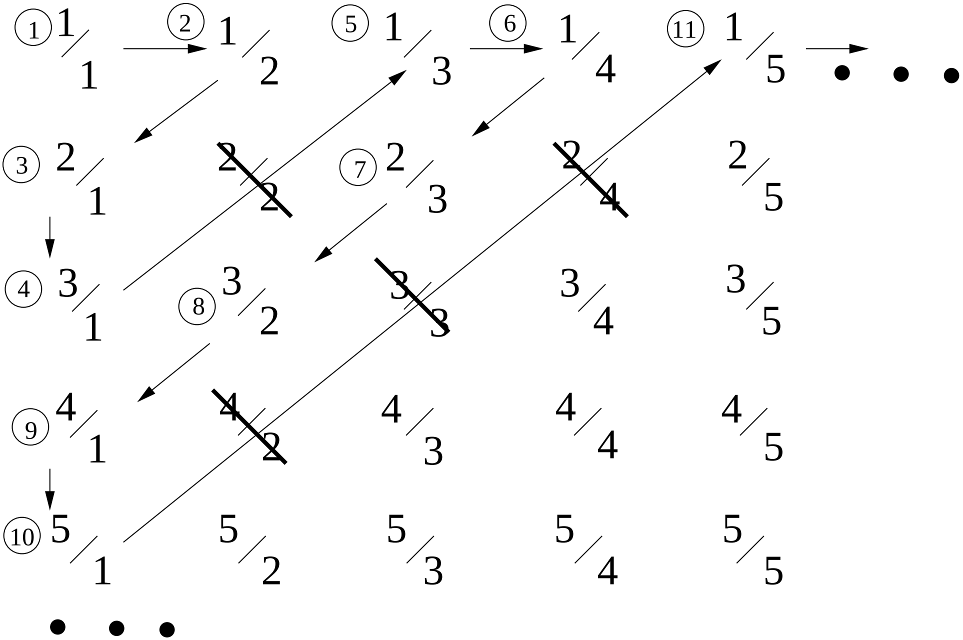
Since f is a one-to-one correspondence, $\mathbb{Z}$ is countably infinite. $\square$

$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	$\cdots$
$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	
$3/1$	$3/2$	$3/3$	$3/4$	$3/5$	
$4/1$	$4/2$	$4/3$	$4/4$	$4/5$	
$5/1$	$5/2$	$5/3$	$5/4$	$5/5$	
$\cdots$					


1/1 1/2 1/3 1/4 1/5
2/1 2/2 2/3 2/4 2/5
3/1 3/2 3/3 3/4 3/5
4/1 4/2 4/3 4/4 4/5
5/1 5/2 5/3 5/4 5/5
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

```
fun findRoom(busNum, seatNum) =
  let
    fun nextPair(a, b) =
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)
      else if b = 1 andalso a mod 2 = 0
           then (a + 1, 1)
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)
      else (a - 1, b + 1);
    fun findRoomHelper(i, currentPair) =
      if currentPair <> (busNum, seatNum)
      then findRoomHelper(i + 1, nextPair(currentPair))
      else i;
  in
    findRoomHelper(1, (1, 1))
end;
```

```
fun findBusSeat(room) =
  let
    fun nextPair(a, b) =
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)
      else if b = 1 andalso a mod 2 = 0
           then (a + 1, 1)
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)
      else (a - 1, b + 1);
    fun findBusSeatHelper(i, currentPair) =
      if i <> room
      then findBusSeatHelper(i + 1,
                             nextPair(currentPair))
      else currentPair;
  in
    findBusSeatHelper(1, (1, 1))
  end;
```



1 1/1
2 1/2
5 1/3
6 1/4
11 1/5
3 2/1
2/2
7 2/3
2/4
2/5
4 3/1
8 3/2
3/3
3/4
3/5
9 4/1
4/2
4/3
4/4
4/5
10 5/1
5/2
5/3
5/4
5/5

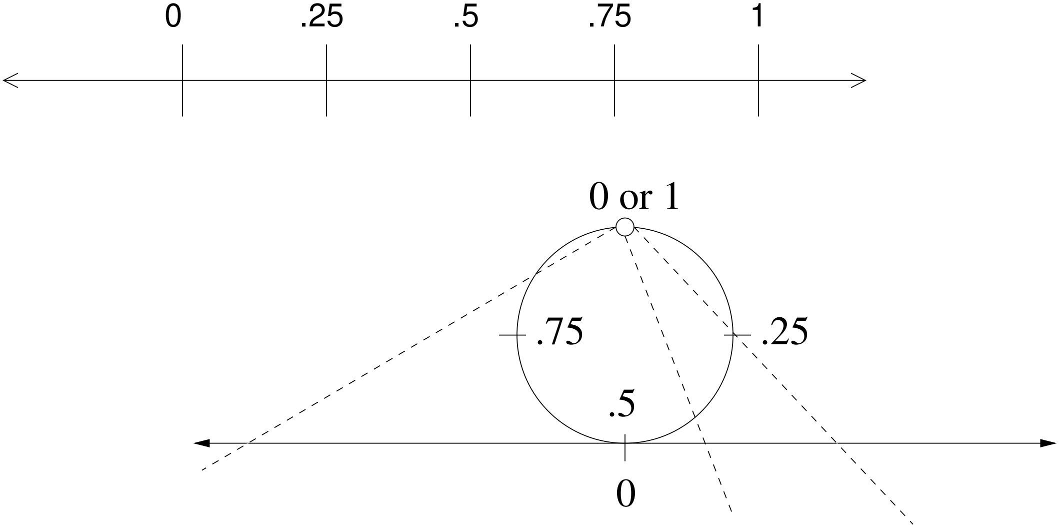
Thm 7.21. $\mathbb{Q}^+$ is countably infinite.

So,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}|$$

What about $\mathbb{R}$?

Thm 7.22. $(0,1)$ has the same cardinality as $\mathbb{R}$.



Thm 7.23. $(0,1)$ is uncountable.

Proof. *Suppose* $(0,1)$ *is countable. Then there exists a one-to-one correspondence* $f : \mathbb{N} \to (0,1)$*. We will use* f *to give names to the all the digits of all the numbers in* $(0,1)$*, considering each number in its decimal expansion, where each* $a_{i,j}$ *stands for a digit.:*

$$\begin{array}{rcl} f(1) & = & 0.a_{1,1}a_{1,2}a_{1,3}\dots a_{1,j}\dots \\ f(2) & = & 0.a_{2,1}a_{2,2}a_{2,3}\dots a_{2,j}\dots \\ & \vdots & \\ f(x) & = & 0.a_{x,1}a_{x,2}a_{x,3}\dots a_{x,j}\dots \\ & \vdots & \end{array}$$

Now construct a number $d = 0.d_1d_2d_3\dots d_i\dots$ *as follows*

$$d_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1 \\ 2 & \text{if } a_{i,i} = 1 \end{cases}$$

Since $d \in (0,1)$ *and* f *is onto, there exists an* $x \in \mathbb{N}$ *such that* $f(x) = d$. *Moreover,*

$$f(x) = 0.a_{x,1}a_{x,2}a_{x,3}\ldots a_{x,x}\ldots$$

so

$$d = 0.a_{x,1}a_{x,2}a_{x,3}\ldots a_{x,x}\ldots$$

by substitution. In other words, $d_i = a_{x,i}$, *and specifically* $d_x = a_{x,x}$. *However, by the way that we have defined* d, *we know that* $d_x \neq a_{x,x}$, *a contradiction. Therefore* $(0,1)$ *is not countable.* $\square$