Chapter 4 roadmap:

- Subset proofs (last week Monday)
- Set equality and emptiness proofs (last week Wednesday)
- Conditional and biconditional proofs (last week Friday)
- Proofs about powersets (**Today**)
- From theorems to algorithms (Friday)
- (Start Chapter 5 next week)

Today: Case study of large proof (powersets)

- Review of powersets and their recursive structure
- Big result
- Warm-up proofs
- Proving the big result

Consider the set $A = \{1, 2, 3, 4, 5\}$. Which of the following is true about the powerset $\mathscr{P}(A)$? (Only one is true.)

$\{3\} \in \mathscr{P}(A)$

$3 \in \mathscr{P}(A)$

$\{3\} \subseteq \mathscr{P}(A)$

$3 \subseteq \mathscr{P}(A)$

$$
\begin{aligned}
A &= \{a, b, c\} & \mathscr{P}(A) &= \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\} \\
& & & \quad \{b, c\}, \{b\}, \{c\}, \emptyset\} \\
A - \{a\} &= \{b, c\} & \mathscr{P}(A - \{a\}) &= \{\{b, c\}, \{b\}, \{c\}, \emptyset\}
\end{aligned}
$$

$$
\{\{a\} \cup C \mid C \in \mathscr{P}(A - \{a\})\} = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}\}
$$

$$
\begin{aligned}
\mathscr{P}(A) &= \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\} &= \{\{a\} \cup C \mid C \in \mathscr{P}(A - \{a\})\} \\
& \quad \{b, c\}, \{b\}, \{c\}, \emptyset\} & \cup \mathscr{P}(A - \{a\})
\end{aligned}
$$

$$A = \{a, b, c\} \quad \mathscr{P}(A - \{a\}) = \{\{b, c\}, \{b\}, \{c\}, \emptyset\}$$

$$\{\{a\} \cup C \mid C \in \mathscr{P}(A - \{a\})\} = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}\}$$

If $a \in A$, then $\mathscr{P}(A)$ consists in $\mathscr{P}(A - \{a\})$ and $\{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\}$

Corollary 4.12. If $a \in A$, then $\mathscr{P}(A - \{a\})$ and $\{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\}$ make a partition of $\mathscr{P}(A)$.

$A \subseteq B$ iff $A \in \mathscr{P}(B)$

$A \in \mathscr{P}(A)$

$\emptyset \in \mathscr{P}(A)$

$a \in A$ iff $\{a\} \in \mathscr{P}(A)$

Warm-up proofs:

Theorem 4.7. If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

Exercise 4.9.1. If $B \subseteq A$, then $\mathscr{P}(B) - \mathscr{P}(A) = \emptyset$.

Roadmap

Corollary 4.12
$\mathscr{P}(A - \{a\})$ and $\{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\}$
make a partition of $\mathscr{P}(A)$.

$\nearrow$ $\nwarrow$

Theorem 4.11 / Exercise 4.9.6
$\mathscr{P}(A - \{a\}) \cap$
$\{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\} = \emptyset$

Theorem 4.10.
$\mathscr{P}(A - \{a\}) \cup \{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\} = \mathscr{P}(A)$

$\nearrow$ $\nwarrow$

Lemma 4.9.
$\mathscr{P}(A - \{a\}) \cup \{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\}$
$\subseteq \mathscr{P}(A)$

Lemma 4.8.
$\mathscr{P}(A) \subseteq$
$\mathscr{P}(A - \{a\}) \cup \{C \cup \{a\} \mid C \in \mathscr{P}(A - \{a\})\}$

For next time:

Pg 174: 4.9.(1, 3, 4, 6)

Skim 4.(10 & 11)