Chapter 5 roadmap:

- Introduction to relations (Monday)
- Properties of relations (**Today** and Friday)
- Transitive closure (after-break Monday)
- Partial order relations (after-break Wednesday)
- Review for Test 2 (after-break Friday)

Today and next time:

- Review of definitions from last time
- Properties of relations
  - Reflexivity
  - Symmetry
  - Transitivity
- Proofs
- More proofs

For next time (Tue, Mar 15):

Pg 205: 5.3.(5, 11, 14)
Pg 208: 5.4.(3, 4, 5, 22, 24, 25)
Pg 212: 5.5.(7, 9, 10)

A **relation** from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A **relation** on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The **image** of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The **image** of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The **inverse** of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The **composition** of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The **identity** relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	$=$

Ex 5.3.7. Prove that if R is a relation on a set A and $(a, b) \in R$, then $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$.

Proof. *Suppose R is a relation over A and that $(a, b) \in R$.*

[Note that $(a, b) \in R$ implies that both a and b must be elements of A.]

Suppose $x \in \mathcal{I}_R(b)$. By definition of image, $(b, x) \in R$. Since $(a, b) \in R$, we have $(a, x) \in R \circ R$ by definition of composition. Moreover $x \in \mathcal{I}_{R \circ R}(a)$ by definition of image.

Therefore $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$ by definition of subset. □

Ex 5.3.9. Prove that if R is a relation from A to B, then $i_B \circ R = R$.

Proof. *First suppose* $(x, y) \in i_B \circ R$. *By definition of composition, there exists* $b \in B$ *such that* $(x, b) \in R$ *and* $(b, y) \in i_B$.

By definition of the identity relation, $b = y$. *By substitution,* $(x, y) \in R$. *Hence* $i_B \circ R \subseteq R$ *by definition of subset.*

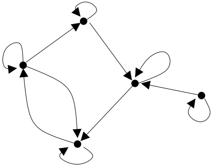
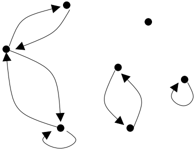
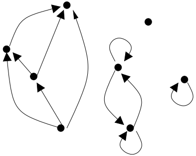
Next suppose $(x, y) \in R$. *By how* R *is defined, we know* $x \in A$ *and* $y \in B$.

By definition of the identity relation, $(y, y) \in i_B$. *By definition of composition,* $(x, y) \in i_B \circ R$. *Hence* $R \subseteq i_B \circ R$.

Therefore, by definition of set equality, $i_B \circ R = R$. □

HW. Ex 5.3.8. Is $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$? Is $A \subseteq \mathcal{I}_{R^{-1}}(\mathcal{I}_R(A))$?

HW. Ex 5.3.10. $(R^{-1})^{-1} = R$.

	Reflexivity	**Symmetry**	**Transitivity**
Informal	Everything is related to itself	All pairs are mutual	Anything reachable by two hops is reachable by one hop
Formal	$\forall\ x \in X,\ (x, x) \in R$	$\forall\ x, y \in X,\ (x, y) \in R \rightarrow (y, x) \in R$ OR $\forall (x, y) \in R,\ (y, x) \in R$	$\forall x, y, z \in X,$ $(x, y), (y, z) \in R \rightarrow (x, z) \in R$ OR $\forall (x, y), (y, z) \in R, (x, z) \in R$
Visual			
Examples	$\subseteq$, $\leq$, $\geq$, $\equiv$, i, isAquaintedWith, waterVerticallyAligned	$\equiv$, isOppositeOf, isOnSameRiver, isAquaintedWith	$<$, $\leq$, $>$, $\geq$, $\subseteq$, isTallerThan, isAncestorOf, isWestOf

	Reflexivity	**Symmetry**	**Transitivity**
Formal	$\forall\, x \in X,\ (x,x) \in R$	$\forall\, x,y \in X,$ $(x,y) \in R \rightarrow (y,x) \in R$ OR $\forall\, (x,y) \in R,\ (y,x) \in R$	$\forall\, x,y,z \in X,$ $(x,y),(y,z) \in R \rightarrow (x,z) \in R$ OR $\forall\, (x,y),(y,z) \in R, (x,z) \in R$
Analytical use	Suppose R is reflexive and $a \in X$. Then $(a,a) \in R$.	Suppose R is symmetric $[a,b \in X]$ and $(a,b) \in R$. Then $(b,a) \in R$	Suppose R is transitive $[a,b,c \in X]$ and $(a,b),(b,c) \in R$. Then $(a,c) \in R$.
Synthetic use	Suppose $a \in X$. ... $(a,a) \in R$. Hence R is reflexive.	Suppose $(a,b) \in R$. ... $(b,a) \in R$. Hence R is symmetric.	Suppose $(a,b),(b,c) \in R$. ... $(a,c) \in R$. Hence R is transitive.

Theorem 5.5. $\mid$ (divides) is reflexive.

Exercise 5.4.2. $\mid$ (divides) is not symmetric.

Theorem 5.6. $R \cap R^{-1}$ is symmetric.

Theorem 5.7. $|$ is transitive.

Exercise 5.4.20. $R^{-1} \circ R$ is reflexive. *(False)*

Exercise 5.4.21. If R and S are both reflexive, then $R \cap S$ is reflexive.

Exercise 5.4.23. If R and S are both symmetric, then $(S \circ R) \cup (R \circ S)$ is symmetric.

Based on Exercise 5.5.5. If R is transitive, then $R \circ R \subseteq R$.

Exercise 5.4.27. If R is transitive, $\mathcal{I}_R(\mathcal{I}_R(A)) \subseteq \mathcal{I}_R(A)$.

Exercise 5.5.4. If R is reflexive and

(for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(c, a) \in R$),

then R is an equivalence relation.

Exercise 5.5.8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation. *(True or false?)*

Exercise 5.5.6. If R is an equivalence relation and $(a, b) \in R$, then $\mathcal{I}_R(a) = \mathcal{I}_R(b)$.

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