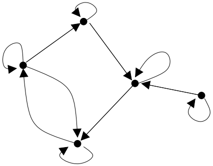
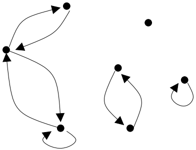
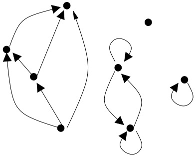
Chapter 5 roadmap:

- Introduction to relations (Monday before break)
- Properties of relations (Wednesday and Friday before break)
- Transitive closure (**Today**)
- Partial order relations (Wednesday)
- Review for Test 2 (Friday)
- Test 2 on Chapters 4 & 5 (next week Monday)

Today:

- Review of relation properties
- An arithmetic on relations
- Computing whether a function is transitive
- Transitive closure

A **relation** from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A **relation** on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The **image** of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The **image** of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The **inverse** of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The **composition** of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The **identity** relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	$=$

	Reflexivity	**Symmetry**	**Transitivity**
Informal	Everything is related to itself	All pairs are mutual	Anything reachable by two hops is reachable by one hop
Formal	$\forall\ x \in X,\ (x, x) \in R$	$\forall\ x, y \in X,\ (x, y) \in R \to (y, x) \in R$ OR $\forall (x, y) \in R,\ (y, x) \in R$	$\forall x, y, z \in X,$ $(x, y), (y, z) \in R \to (x, z) \in R$ OR $\forall (x, y), (y, z) \in R, (x, z) \in R$
Visual			
Examples	$\subseteq$, $\leq$, $\geq$, $\equiv$, i, isAquaintedWith, waterVerticallyAligned	$\equiv$, isOppositeOf, isOnSameRiver, isAquaintedWith	$<$, $\leq$, $>$, $\geq$, $\subseteq$, isTallerThan, isAncestorOf, isWestOf

Operators	$x + y$	$p \vee q$	$A \cup B$
	$-x$	$\sim p$	$\overline{A}$
Distribution	$x \cdot (y + z)$	$p \wedge (q \vee r)$	$A \cap (B \cup C)$
	$= x \cdot y + x \cdot z$	$\equiv (p \wedge q) \vee (p \wedge r)$	$= (A \cap B) \cup (A \cap C)$
Identity	$x + 0 = x$	$p \vee T \equiv p$	$A \cup \emptyset = A$
	$x \cdot 1 = x$	$p \wedge F \equiv p$	$A \cap \mathcal{U} = A$

$S \circ R$

R^{-1}

$i_X \circ R = R$

$R^2 = R \circ R$

R	is one less than	eats	is parent of
R^2	is two less than	eats something that eats	is grandparent of
R^3	is three less than	eats something that eats something that eats	is great grandparent of
???	$<$	gets nutrients from	is ancestor of

Definition of transitivity

Short form: $\forall\ (x, y), (y, z) \in R, (x, z) \in R$

Transform this to:

$$\forall\ (x, y) \in R,\ \ \forall\ (w, z) \in R,\ \ \text{if}\ \ y = w\ \ \text{then}\ \ (x, z) \in R$$

Definition of transitivity

Short form: $\forall\ (x, y), (y, z) \in R, (x, y) \in R$

Transform this to:

$$\forall\ (x, y) \in R,\ \ \forall\ (w, z) \in R,\ \ \text{if } \boxed{y = w} \text{ then } \boxed{(x, z) \in R}$$

Definition of transitivity

Short form: $\forall\ (x,y),(y,z) \in R, (x,y) \in R$

Transform this to:

$$\forall\ (x,y) \in R,\ \ \forall\ (w,z) \in R,\ \boxed{\text{if } \boxed{y = w} \text{ then } \boxed{(x,z) \in R}}$$

Definition of transitivity

Short form: $\forall\ (x, y), (y, z) \in R, (x, y) \in R$

Transform this to:

$$\forall\ (x, y) \in R,\ \boxed{\forall\ (w, z) \in R,\ \boxed{\text{if } \boxed{y = w} \text{ then } \boxed{(x, z) \in R}}}$$

Definition of transitivity

Short form: $\forall\ (x, y), (y, z) \in R, (x, y) \in R$

Transform this to:

$$\boxed{\forall\ (x, y) \in R,\ \boxed{\forall\ (w, z) \in R,\ \boxed{\text{if } \boxed{y = w} \text{ then } \boxed{(x, z) \in R}}}}$$

$\{(1, 2), (2, 3), (5, 2), (1, 5), (2, 5), (1, 3)\}$

$\{(1, \downarrow 2), (2, 3), (5, 2), (1, 5), (2, 5), (1, 3)\}$

$\{(1, 2), (2, \downarrow 3), (5, 2), (1, 5), (2, 5), (1, 3)\}$

$\{(1, 2), (2, 3), (5, \downarrow 2), (1, 5), (2, 5), (1, 3)\}$

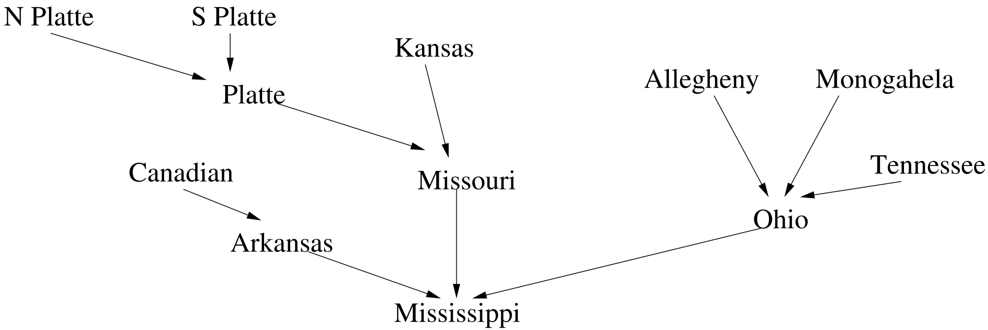
Computing transitivity is a $\forall\ \forall\ \exists$ problem

Our strategy is, for each pair (x, y), walk through the whole (original) list. If the list

1. is empty, then true (vacuously)
2. begins with (y, z) (that is, begins with (w, z) where $y = w$), then search the whole (original) list for (x, z).
   2.1 if found, keep searching
   2.2 if not found, then false
3. begins with (w, z) for $w \neq y$, skip it and keep searching

Domain	First relation	Second relation
Rivers	*flows into* The Platte flows into the Missouri, and the Missouri flows into the Mississippi.	*is tributary to* The Platte is a tributary to the Missouri; both the Platte and the Missouri are tributaries to the Mississippi.
People	*is parent of* Bill is Jane's parent; Jane is Leroy's parent	*is ancestor of* Bill is Jane's ancestor; Leroy has both Jane and Bill as ancestors.

Domain	**First relation**	**Second relation**
Animals	*eats*	*derives nutrients from*
	Rabbit eats clover; coyote eats rabbit.	Coyote derives nutrients from rabbit; rabbit derives nutrients from clover; both coyote and rabbit ultimately derive nutrients from clover.
$\mathbb{Z}$	*is one less than*	$<$
	2 is one less than 3; 3 is one less than 4	$2 < 3$; $3 < 4$; $2 < 4$.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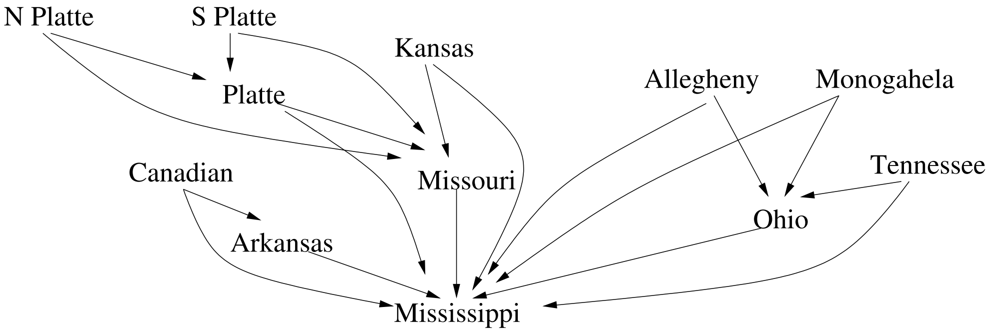
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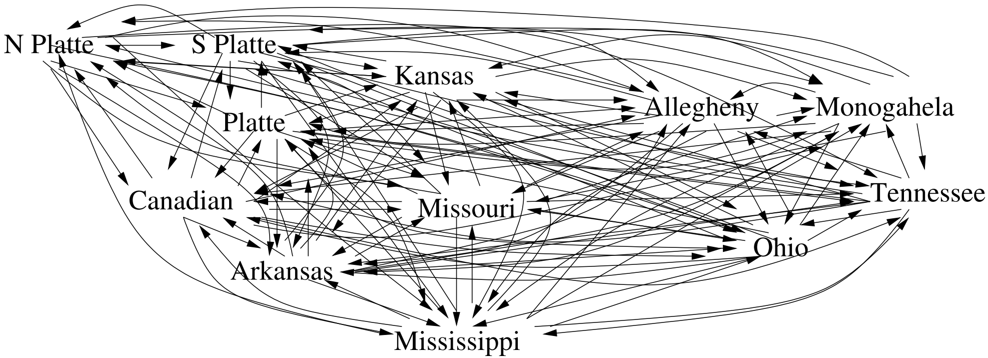
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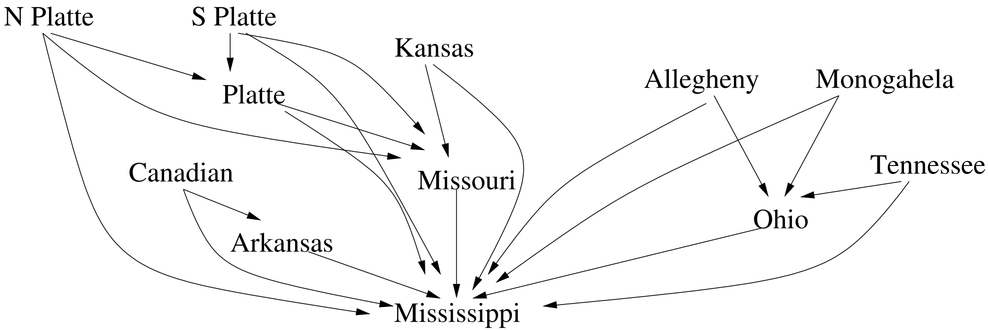
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Ohio
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Mississippi

If R is a relation on X, then R^T is the **transitive closure** of R if

- R^T is transitive
- $R \subseteq R^T$
- If S is a transitive relation such that $R \subseteq S$, then $R^T \subseteq S$

Theorem 5.12 *The transitive closure of a relation R is unique.*

Proof. *Suppose S and T are relations fulfilling the requirements for being transitive closures of R. By items 1 and 2, S is transitive and $R \subseteq S$, so by item 3, $T \subseteq S$. By items 1 and 2, T is transitive and $R \subseteq T$, so by item 3, $S \subseteq T$. Therefore $S = T$ by the definition of set equality.* □

Other closures:

Ex 5.7.2 $R \cup i_A$ is the reflexive closure of R

Ex 5.7.3. $R \cup R^{-1}$ is the symmetric closure of R. (HW)

Ex 5.7.2 $R \cup i_A$ is the reflexive closure of R

Proof. *Suppose R is a relation on A.*

[$R \cup i_A$ is reflexive:] Suppose $a \in A$. $(a, a) \in i_A$ by definition of identity relation. $(a, a) \in R \cup i_A$ by definition of union. Hence $R \cup i_A$ is reflexive by definition.

[$R \subseteq R \cup i_A$:] Suppose $(a, b) \in R$. Then $(a, b) \in R \cup i_A$ by definition of uniion. Hence $R \subseteq R \cup i_A$. (Alternately, we could have cited Exercise 4.2.1.)

[$R \cup i_A$ is the smallest such relation:] Suppose S is a reflexive relation such that $R \subseteq S$. Suppose further $(a, b) \in R \cup i_A$. By definition of union, $(a, b) \in R$ or $(a, b) \in i_A$.

Case 1: *Suppose $(a, b) \in R$. Then $(a, b) \in S$ by definition of subset (since we supposed $R \subseteq S$).*

Case 2: *Suppose $(a, b) \in i_A$. Then, by definition of identity relation, $a = b$. $(a, a) \in S$ by definition of reflexive (since we suppose S is reflexive). $(a, b) \in S$ by substitution.*

Either way, $(a, b) \in S$ and hence $R \cup i_A \subseteq S$ by definition of subset.

Therefore, $R \cup i_A$ is the reflexive closure of R. □

Theorem 5.13 *If R is a relation on a set A, then*

$$R^{\infty} = \bigcup_{i=1}^{\infty} R^i = \{(x,y) \mid \exists\ i \in \mathbb{N} \text{ such that } (x,y) \in R^i\}$$

is the transitive closure of R.

Proof. *Suppose R is a relation on a set A.*
Suppose $a, b, c \in A$, $(a,b), (b,c) \in R^{\infty}$. By the definition of R^{∞}, there exist $i, j \in \mathbb{N}$ such that $(a,b) \in R^i$ and $(b,c) \in R^j$. By the definition of relation composition and Exercise 5.7.4, $(a,c) \in R^j \circ R^i = R^{i+j}$. $R^{i+j} \subseteq R^{\infty}$ by the definition of R^{∞}. By the definition of subset, $(a,c) \in R^{\infty}$. Hence, R^{∞} is transitive by definition.
Suppose $a, b \in A$ and $(a,b) \in R$. By the definition of R^{∞} (taking $i = 1$), $(a,b) \in R^{\infty}$, and so $R \subseteq R^{\infty}$, by definition of subset.
Suppose S is a transitive relation on A and $R \subseteq S$. Further suppose $(a,b) \in R^{\infty}$. Then, by definition of R^{∞}, there exists $i \in \mathbb{N}$ such that $(a,b) \in R^i$. By Lemma 5.14, $(a,b) \in S$. Hence $R^{\infty} \subseteq S$ by definition of subset.
Therefore, R^{∞} is the transitive closure of R. $\square$

For next time:

Pg 217: 5.6.(1 & 3)
Pg 222: 5.7.(3,4,5)

For Exercise 5.7.4, it should say $(S \circ R) \circ Q = S \circ (R \circ Q)$ *instead of* $(R \circ S) \circ Q = R \circ (S \circ Q)$.

Read 5.(8 & 9)