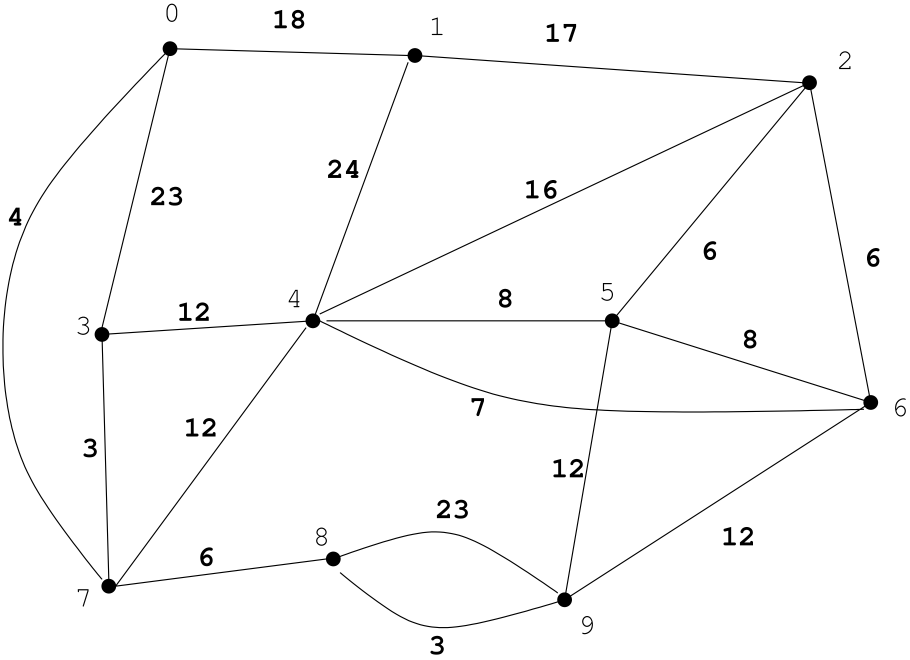
Chapter 4, Graphs:

- Concepts and implementation (Friday, Feb 11)
- Traversal (Monday, Feb 14)
- Minimum spanning trees (**Wednesday and Friday**)
- Single-source shortest paths (next week Monday)
- (Start BSTs Wednesday Mar 2)

"Today" (Wednesday and Friday):

- MST problem definition
- Brute-force solution
- General structure of good solutions
- Kruskal's algorithm, plus proof and analysis
- Prim's algorithm, plus proof and analysis
- Performance comparison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**General strategy for MST** (both algorithms):

- Maintain a set of edges $A$ that is a subset of a MST
- At each step, add one edge to $A$ until it's a MST

## Invariant (General MST main loop)

*There exists $T \subseteq E$ such that $T$ is a minimum spanning tree of $G$ and $A \subseteq T$.*

**General algorithm outline:**

$A = \emptyset$
While $A$ isn't a MST
    add an edge to $A$ that maintains the invariant

**Insight 1:** $A$ implicitly partitions vertices into connected components. The lightest edge that connects two components is safe.

Lemma (Safe edges in Kruskal's algorithm.)

*If $G = (V, E)$ is a graph, $A$ is a subset of a minimum spanning tree for $G$, and $(u, v)$ is the lightest edge connecting any distinct connected components of $A$, then $(u, v)$ is a safe edge for $A$, that is, $A \cup \{(u, v)\}$ is a subset of a minimum spanning tree.*

**Proof.** Suppose everything in the hypothesis, in particular that $A$ is a subset of some minimum spanning tree $T$ and that $u$ and $v$ are in distinct connected components of $A$, call them $A_u$ and $A_v$. Let $w_T$ be the total weight of $T$, that is, the sum of the weights of all the edges of $T$. We want to prove that adding $(u, v)$ to $A$ makes something that is still a subset of some minimum spanning tree.

If $(u, v) \in T$, then we're done. Suppose, then, that $T$ does not contain $(u, v)$. Since $T$ is a spanning tree, it means that $u$ and $v$ are connected in $T$. Pick the lightest edge on the path from $u$ to $v$ that is not in $A$, call it $(x, y)$. Essentially $(x, y)$ is an edge that was picked instead of $(u, v)$ that contributed to connecting $A_u$ and $A_v$.

Snip out $(x, y)$. This would disconnect $T$, that is, the graph $T - \{(x, y)\}$ is not a tree, but rather contains two connected components, one with $u$ in it and the other with $v$ in it. Now splice in $(u, v)$. That will reconnect $u$ and $v$ and make it into a tree again. Formally we've made a new spanning tree $(T - \{(x, y)\}) \cup \{(u, v)\}$.

The hypothesis says that $(u, v)$ was the lightest edge connecting distinct components of $A$. That means $w(u, v) \leq w(x, y)$. That in turn means that the total weight of the new spanning tree is also just as good, if not better, than the old one: $w_{T-\{(x,y)\})\cup\{(u,v)\}} \leq w_T$. Since it ties or beats a (supposed) minimum spanning tree, $(T - \{(x, y)\}) \cup \{(u, v)\}$ must be a minimum spanning tree. Therefore $(u, v)$ is safe. □

| | | Kruskal | | Prim | |
|---|---|---|---|---|---|
| | | Unoptimized | Optimized | Unoptimized | Optimized |
| Sparse | Adjacency list | 31579 | 28841 | 72364 | 58089 |
| | Adjacency matrix | 49128 | 35493 | 67887 | 49537 |
| Medium | Adjacency list | 147527 | 54877 | 180407 | 113555 |
| | Adjacency matrix | 127485 | 59821 | 146358 | 75906 |
| Dense | Adjacency list | 136762 | 69867 | 191617 | 123762 |
| | Adjacency matrix | 162468 | 78154 | 130984 | 72245 |