Chapter 5, Binary search trees:

- Binary search trees; the balanced BST problem (spring-break eve; finishing **Today**)
- AVL trees (**Today** and Wednesday)
- Traditional red-black trees (Friday)
- Left-leaning red-black trees (next week Monday)
- "Wrap-up" BST (next week Wednesday)

Today and Monday:

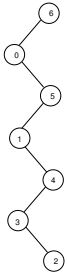
- Review BST basics
- BST performance and the balanced BST problem
- Introduction to the code base
- AVL tree definition
- AVL tree cases
- AVL tree performance

A **binary search tree** (BST) over some ordered key type is either

- empty, or
- a node augmented with a key $k$ together with two children, designated *left* and *right*, such that
  - *left* is a binary search tree such that all of the keys in that tree are less than or equal to $k$, and
  - *right* is a binary search tree such that all of the keys in that tree are greater than or equal to $k$.

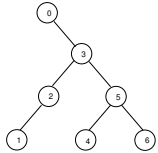
| | | Unsorted | Sorted |
|---|---|---|---|
| Array | Find | $\Theta(n)$ | $\Theta(\lg n)$ |
| | Insert | $\Theta(1)$ expected, $\Theta(n)$ worst | $\Theta(n)$ |
| | Delete | $\Theta(n)$ | $\Theta(n)$ |
| Linked structure | Find | $\Theta(n)$ | $\Theta(n)$ |
| | Insert | $\Theta(1)$ | $\Theta(1)$ |
| | Delete | $\Theta(1)$ | $\Theta(1)$ |

6, 0, 5, 1, 4, 2, 3



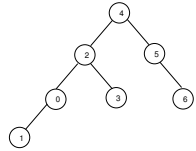

height 7
total depth 21
ANI 4

0, 3, 5, 2, 6, 1, 4



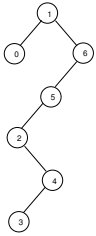

height 4
total depth 14
ANI 3

4, 2, 5, 3, 0, 1, 6



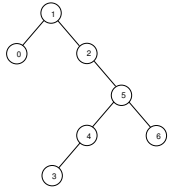

height 4
total depth 11
ANI 2.57

1, 6, 5, 2, 4, 3, 0



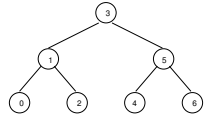

height 6
total depth 16
ANI 3.29

1, 2, 5, 4, 3, 0, 6

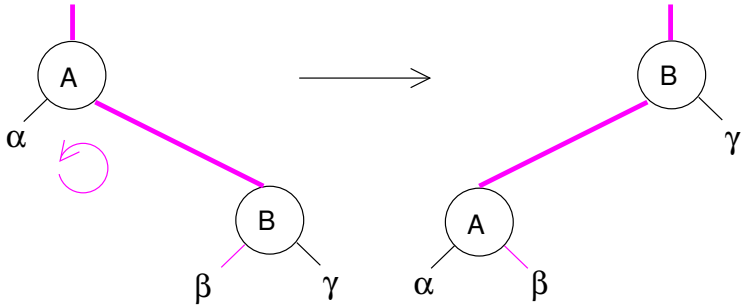



height 5
total depth 14
ANI 3

3, 1, 5, 0, 2, 4, 6




height 3
total depth 10
ANI 2.43


A
α
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B
γ
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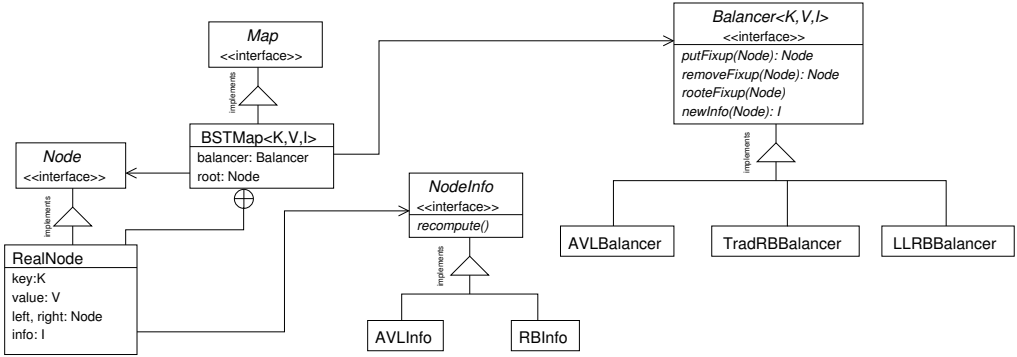
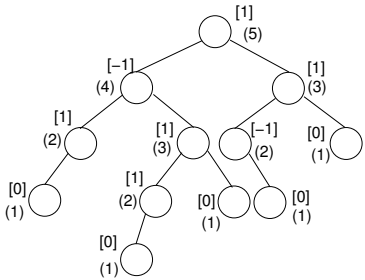
α
β


Map
<<interface>>
implements
BSTMap<K,V,I>
balancer: Balancer
root: Node
Node
<<interface>>
implements
RealNode
key:K
value: V
left, right: Node
info: I
NodeInfo
<<interface>>
recompute()
implements
AVLInfo
RBInfo
Balancer<K,V,I>
<<interface>>
putFixup(Node): Node
removeFixup(Node): Node
rooteFixup(Node)
newInfo(Node): I
implements
AVLBalancer
TradRBBalancer
LLRBBalancer


[1]
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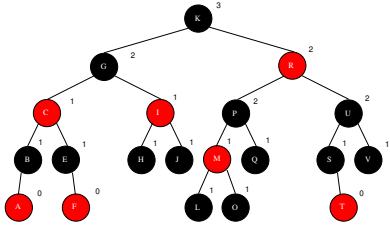
K
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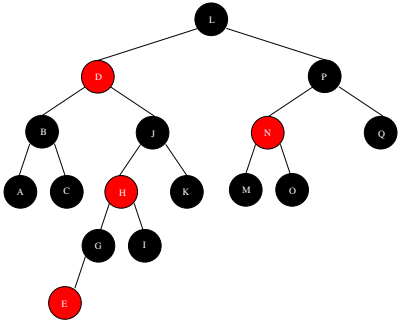
L
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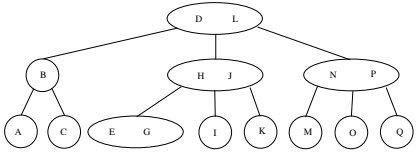
D L
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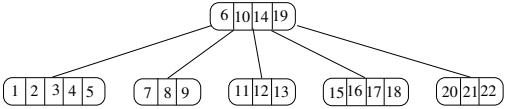
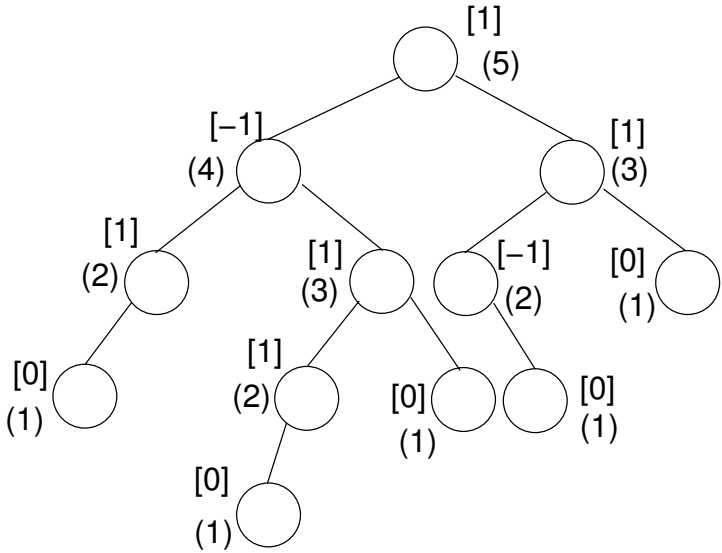

6 10 14 19
1 2 3 4 5
7 8 9
11 12 13
15 16 17 18
20 21 22


[1]
(5)
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(2)
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(3)
[−1]
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[0]
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[0]
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[1]
(2)
[0]
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[0]
(1)
[0]
(1)

The *height* of a node (or (sub)tree) is the number of nodes on any path from that node to any leaf, inclusive.

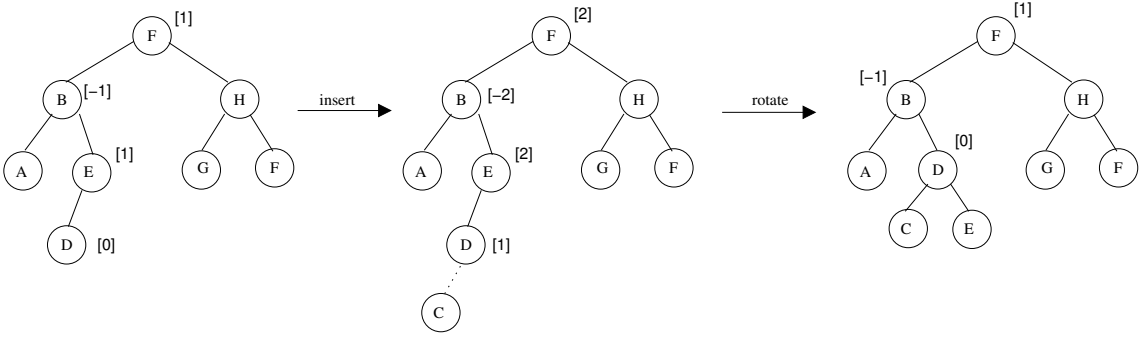
$$height(c) = \begin{cases} 0 & \text{if } c \text{ is null} \\ \max(height(c.\ell) + height(c.r)) + 1 & \text{otherwise} \end{cases}$$

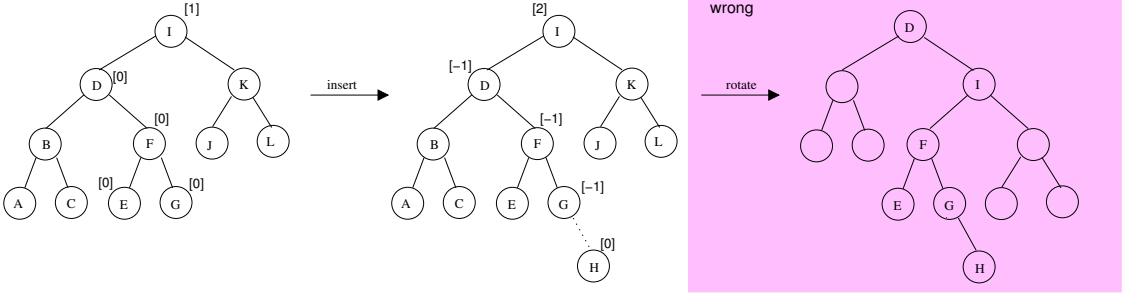
The *balance* of a node is the difference between the heights of its left and right children. In an AVL tree, each node's subtrees' heights must differ by at most 1:

$$\forall\ x \in \text{nodes}, |height(x.\text{left}) - height(x.\text{right})| \leq 1$$

A node that has balance 1 or -1 has a *bias*. A node that (temporarily) has balance 2 or -2 is in *violation*.

(A balance less than -2 or greater than 2 shouldn't happen even temporarily.)


F
[1]
B
[−1]
H
A
E
[1]
G
F
D
[0]
insert
F
[2]
B
[−2]
H
A
E
[2]
G
F
D
[1]
C
rotate
[1]
F
[−1]
B
H
A
D
[0]
G
F
C
E


[1]
I
D
[0]
K
B
[0]
F
J
L
A
C
[0]
E
G
[0]
insert
[2]
I
[−1]
D
K
B
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J
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E
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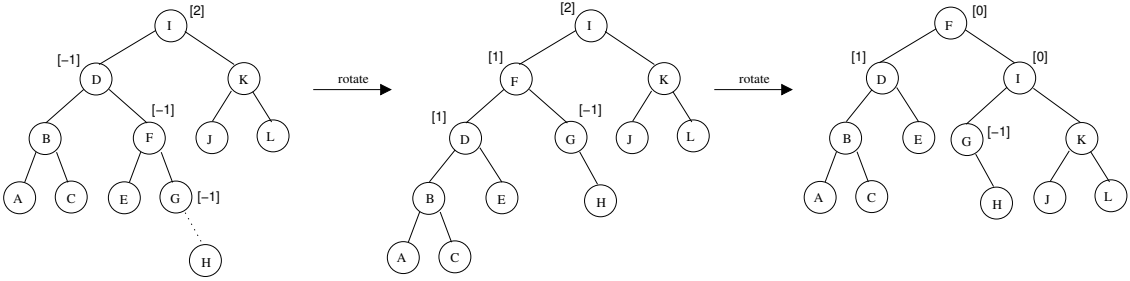
[−1]
[0]
H
wrong
rotate
D
I
F
E
G
H


I
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[−1]
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rotate
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[2]
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[1]
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[1]
G
[−1]
J
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A
C
rotate
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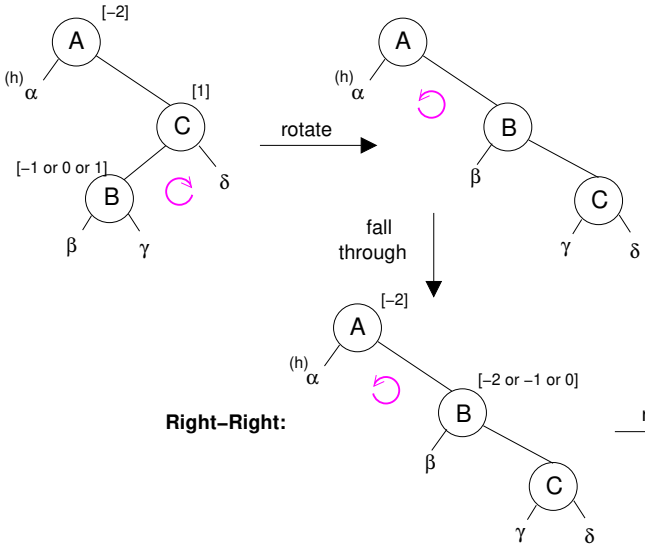
**Right–Left:**




**Right–Right:**

**Invariant 30 (Postconditions of `RealNode.put()` with `AVLBalancer`.)**
Let $x$ be the root of a subtree on which `put()` is called and $y$ be the node returned, that is, the root of the resulting subtree. The subtree rooted at $y$ has no violations and the height of the subtree rooted at $y$ is equal to or one greater than the original height of the subtree rooted at $x$.

> **Proof.** *Suppose `put()` is called on node $x$ in a BST using AVL balancing which has no violations. There are three cases: $x$ is `nully`, $x$ is a `RealNode` containing the key being searched for, or $x$ is a `RealNode` with a different key. We use structural induction with the first two cases as base cases.*

**Base case 1.** *Suppose $x$ is `nully`, which has height 0 Then the node $y$ returned is a new `RealNode` with `nully` as both children, height 1, and balance 0. The subtree rooted at $y$ has no violations and height one greater than the original height of $x$.*

**Base case 2.** *Suppose $x$ is a `RealNode` whose key is equal to the key used for this `put()`. Then the value at node $x$ is overwritten but node $x$ itself is returned (so $y = x$ in this case) with the tree structure unchanged.*

**Inductive case.** *Suppose $x$ is a `RealNode` and, without loss of generality, the key used for this `put()` is greater than the key at $x$, and so `put()` is called on the right child of $x$. Let $h_0$ be the height of the tree rooted at $x$ before this call to `put()` on the right child, and let $z$ the root of the subtree that results from this call to `put()` on the right child. Our inductive hypothesis is that the subtree rooted at $z$ has no violations and that its height is equal to or one greater than the height of the original right child of $x$.*

*Let $h_1$ be the height of the tree rooted at $x$ after the call to `put()` on the right child but before the call to `putFixup()` with $x$.*

*Since since at most the height of its right subtree has increased by one, either $h_1 = h_0$ or $h_1 = h_0 + 1$. By supposition, the balance of $x$ before the call to `put()` was no less than $-1$, since we supposed the tree had no violations. Since at most the height of its right subtree has increased by one, the balance of $x$ is now no less than $-2$. We now have two subcases: Either the balance of $x$ is greater than $-2$ or it is equal to $-2$.*

*Suppose the balance of $x$ is greater than $-2$. Then the call to `putFixup()` with $x$ returns $x$ unchanged, which is also returned as the result of `put()` (again $y = x$), a tree with no violations and height $h_1$.*

*On the other hand, suppose the balance of $x$ is equal to $-2$. Then $y$ is a node other than $x$ returned by `putFixup()`. Let $h_2$ be the height of the subtree rooted at $y$ when `putFixup()` returns. By inspection of the right-right and right-left subcases given above, the subtree rooted at $y$ has no violations and either $h_2 = h_1$ or $h_2 = h_1 - 1$. In either of those cases $h_2 = h_0$ or $h_2 = h_0 + 1$.*
□

**Coming up:**

*Do* **BST rotations** *project (suggested by Wednesday, Mar 16)*

*Due* **Tues, Mar 15** *(end of day)*
*Read Section 5.3*
*Do Exercises 5.(7 & 8)*
*Take quiz*

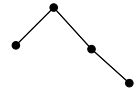
*Do* **AVL** *project (suggested by Monday, Mar 212)*

*Due* **Wed, Mar 23** *(end of day) (but spread it out)*
*Read Sections 5.(4-6) [some parts carefully, some parts skim, some parts optional—see Schoology]*
*Do Exercise 5.14*
*Take quiz*

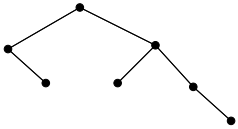
$A_1$

$A_2$

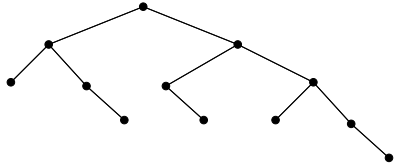
$A_3$



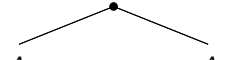
$A_4$



$A_5$



$A_h$



$$B_h = \begin{cases} 1 & \text{if } h = 1 \\ 2 & \text{if } h = 2 \\ B_{h-2} + B_{h-1} + 1 & \text{otherwise} \end{cases} \qquad B_h+1 = \begin{cases} 2 & \text{if } h = 1 \\ 3 & \text{if } h = 2 \\ (B_{h-2} + 1) + (B_{h-1} + 1) & \text{otherwise} \end{cases}$$

| $h$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $B_h + 1$ | 2 | 3 | 5 | 8 | 13 | 21 |
| $B_h$ | 1 | 2 | 4 | 7 | 12 | 20 |

$$
\begin{aligned}
B_h + 1 &> \frac{\phi^{h+2}}{\sqrt{5}} - 1 \\
B_h + 2 &> \frac{\phi^{h+2}}{\sqrt{5}} \\
\sqrt{5}(B_h + 2) &> \phi^{h+2} \\
h + 2 &< \log_\phi(\sqrt{5} B_h) \\
h &< \log_\phi(\sqrt{5} B_h) - 2 \\
&= \log_\phi B_h + \log_\phi \sqrt{5} - 2 \\
&= \frac{1}{\lg \phi} \lg B_h + \log_\phi \sqrt{5} - 2
\end{aligned}
$$