

## Chapter 7 outline:

- ▶ Introduction, function equality, and anonymous functions (Mon, Apr 3)
- ▶ Image and inverse images (Wed, Apr 5)
- ▶ Function properties, applications to programming (last Monday)
- ▶ Function composition (last week Wednesday)
- ▶ Cardinality (last week Friday)
- ▶ *Practice quiz* and Countability (**Today**)
- ▶ Review (Wednesday)
- ▶ Test 3, on Ch 6 & 7 (Friday)

Two finite sets  $X$  and  $Y$  have the *the same cardinality* as each other if there exists a one-to-one correspondence from  $X$  to  $Y$ .

To use this *analytically*:

Suppose  $X$  and  $Y$  have the same cardinality. Then let  $f$  be a one-to-one correspondence from  $X$  to  $Y$ .

$f$  is both onto and one-to-one.

To use this *synthetically*:

Given sets  $X$  and  $Y \dots$

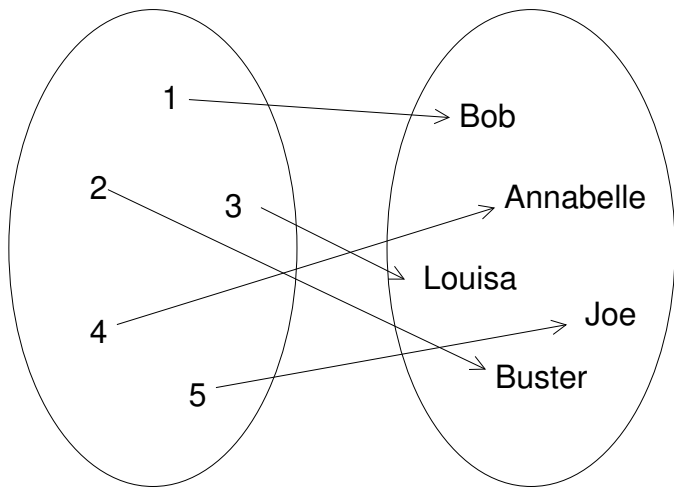
**[Define  $f$ ]** Let  $f : X \rightarrow Y$  be a function defined as  $\dots$

Suppose  $y \in Y$ . *Somehow find*  $x \in X$  such that  $f(x) = y$ . Hence  $f$  is onto.

Suppose  $x_1, x_2 \in X$  such that  $f(x_1) = f(x_2)$ . *Somehow show*  $x_1 = x_2$ . Hence  $f$  is one-to-one.

Since  $f$  is a one-to-one correspondence,  $X$  and  $Y$  have the same cardinality as each other.

A finite set  $X$  has cardinality  $n \in \mathbb{N}$ , which we write as  $|X| = n$ , if there exists a one-to-one correspondence from  $\{1, 2, \dots, n\}$  to  $X$ . Moreover,  $|\emptyset| = 0$ .



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Given a set  $X$ , if there exists  $n \in \mathbb{N}$  and a one-to-one correspondence from  $\{1, 2, \dots, n\}$  to  $X$ , then  $X$  is *finite* and  $|X| = n$ . Otherwise,  $X$  is *infinite*.

Are all infinities equal?

Which is more intuitive to you,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{R}|$$

or

$$|\mathbb{N}| < |\mathbb{W}| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|$$

?

**Thm 7.19.**  $\mathbb{W}$  and  $\mathbb{N}$  have the same cardinality.

**Proof.** [We need a one-to-one correspondence from  $\mathbb{N}$  to  $\mathbb{W}$ .]

Let  $f : \mathbb{W} \rightarrow \mathbb{N}$  be defined so that  $f(n) = n + 1$ .

Suppose  $n \in \mathbb{N}$ . Then  $f(n - 1) = (n - 1) + 1 = n$ , so  $f$  is onto.

Next suppose  $n_1, n_2 \in \mathbb{N}$  such that  $f(n_1) = f(n_2)$ . Then  $n_1 + 1 = n_2 + 1$ , and moreover  $n_1 = n_2$ . Hence  $f$  is one-to-one.

Since a one-to-one correspondence exists between  $\mathbb{W}$  and  $\mathbb{N}$ , the two sets have the same cardinality.  $\square$

A set  $X$  is *countably infinite* if there exists a one-to-one correspondence from  $\mathbb{N}$  to  $X$ .

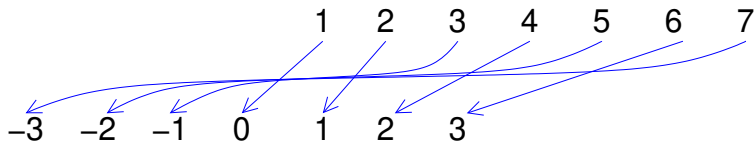
A set is *countable* if it is finite or countably infinite. Otherwise it is *uncountable*.

**Thm 7.20.**  $\mathbb{Z}$  is countably infinite.

**Proof.** [We need a one-to-one correspondence from  $\mathbb{N}$  to  $\mathbb{Z}$ .]

Let  $f : \mathbb{N} \rightarrow \mathbb{Z}$  be defined so that

$$f(x) = \begin{cases} n \operatorname{div} 2 & \text{if } n \text{ is even} \\ -(n \operatorname{div} 2) & \text{otherwise} \end{cases}$$



Since  $f$  is a one-to-one correspondence,  $\mathbb{Z}$  is countably infinite.  $\square$

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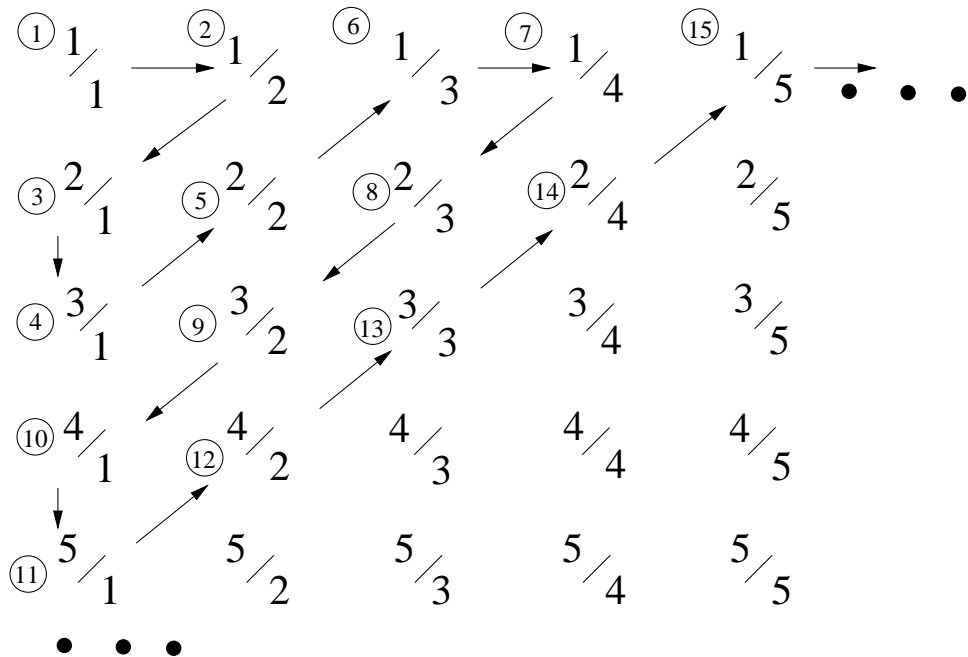
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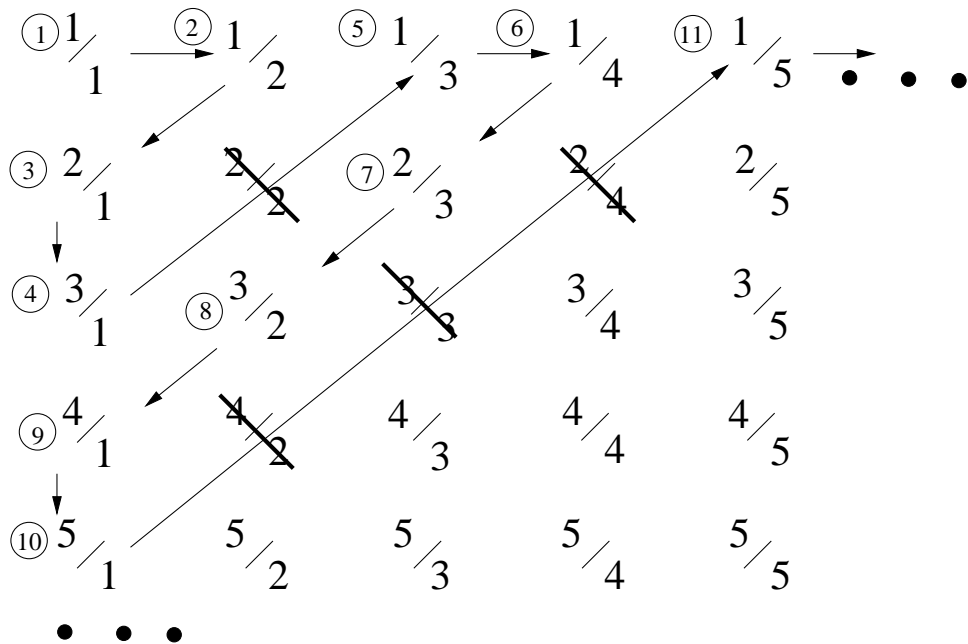
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```
fun findRoom(busNum, seatNum) =  
  let  
    fun nextPair(a, b) =  
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)  
      else if b = 1 andalso a mod 2 = 0  
          then (a + 1, 1)  
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)  
      else (a - 1, b + 1);  
    fun findRoomHelper(i, currentPair) =  
      if currentPair <> (busNum, seatNum)  
      then findRoomHelper(i + 1, nextPair(currentPair))  
      else i;  
  in  
    findRoomHelper(1, (1, 1))  
  end;
```

```
fun findBusSeat(room) =  
  let  
    fun nextPair(a, b) =  
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)  
      else if b = 1 andalso a mod 2 = 0  
          then (a + 1, 1)  
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)  
      else (a - 1, b + 1);  
    fun findBusSeatHelper(i, currentPair) =  
      if i <> room  
      then findBusSeatHelper(i + 1,  
                             nextPair(currentPair))  
      else currentPair;  
  in  
    findBusSeatHelper(1, (1, 1))  
  end;
```



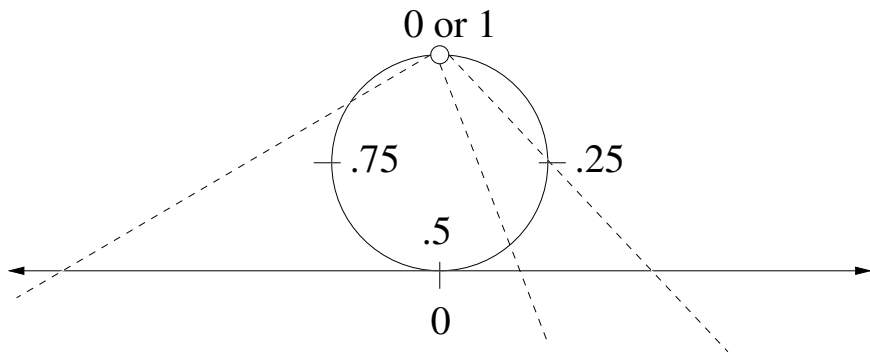
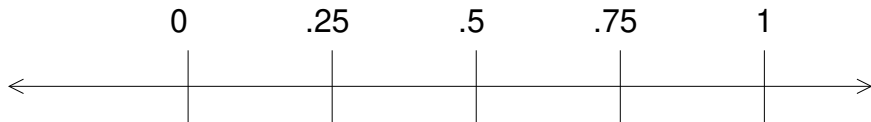
**Thm 7.21.**  $\mathbb{Q}^+$  is countably infinite.

So,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}|$$

What about  $\mathbb{R}$ ?

**Thm 7.22.**  $(0, 1)$  has the same cardinality as  $\mathbb{R}$ .



**Thm 7.23.**  $(0, 1)$  is uncountable.

**Proof.** *Suppose  $(0, 1)$  is countable. Then there exists a one-to-one correspondence  $f : \mathbb{N} \rightarrow (0, 1)$ . We will use  $f$  to give names to all the digits of all the numbers in  $(0, 1)$ , considering each number in its decimal expansion, where each  $a_{i,j}$  stands for a digit.:*

$$\begin{aligned} f(1) &= 0.a_{1,1}a_{1,2}a_{1,3}\dots a_{1,j}\dots \\ f(2) &= 0.a_{2,1}a_{2,2}a_{2,3}\dots a_{2,j}\dots \\ &\vdots \\ f(x) &= 0.a_{x,1}a_{x,2}a_{x,3}\dots a_{x,j}\dots \\ &\vdots \end{aligned}$$

Now construct a number  $d = 0.d_1d_2d_3\dots d_i\dots$  as follows

$$d_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1 \\ 2 & \text{if } a_{i,i} = 1 \end{cases}$$

Since  $d \in (0, 1)$  and  $f$  is onto, there exists an  $x \in \mathbb{N}$  such that  $f(x) = d$ .  
Moreover,

$$f(x) = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

so

$$d = 0.a_{x,1}a_{x,2}a_{x,3} \dots a_{x,x} \dots$$

by substitution. In other words,  $d_i = a_{x,i}$ , and specifically  $d_x = a_{x,x}$ . However, by the way that we have defined  $d$ , we know that  $d_x \neq a_{x,x}$ , a contradiction. Therefore  $(0, 1)$  is not countable.  $\square$