So far, we have seen

- Defining types and sets recursively.
- Proving propositions quantified over recursively defined sets using structural induction.
- Proving propositions quantified over $\mathbb{W}$ or $\mathbb{N}$ using mathematical induction. Specifically, to prove $\forall\ n \in \mathbb{W},\ I(n)$,
  - Prove $I(0)$
  - Prove $\forall\ n \in \mathbb{W},\ I(n) \rightarrow I(n+1)$

Today and Wednesday are about

- Proving the correctness of algorithms using mathematical induction

For next time:

Take quiz (on loop invariants)

For Monday, Apr 3:

Pg 306: 6.10.(2-5)

Read 7 intro and 7.1 carefully
Read 7.2
Skim 7.3
Take quiz (on function introduction)

For any full binary tree T, $\texttt{nodes}(T)$ id odd.

Proof. *By induction on the structure of T.*

Base case. *Suppose T is a leaf. Then $\texttt{nodes}(T) = 1$ by the definition of nodes. Moreover, $\texttt{nodes}(T) = 1 = 2 \cdot 0 + 1$, and so $\texttt{node}(T)$ is odd by definition.*

Inductive case. *Suppose T is an internal node with children T_1 and T_2 such that $\texttt{nodes}(T_1)$ and $\texttt{nodes}(T_2)$ are each odd.*

[By definition of odd, there exist x and y such that $\texttt{nodes}(T_1) = 2x + 1$ and $\texttt{nodes}(T_2) = 2y + 1$.]

Then,

$$\begin{array}{rcll} \texttt{nodes}(T) & = & 1 + \texttt{nodes}(T_1) + \texttt{nodes}(T_2) & \text{By the definition of nodes} \\ & = & 1 + 2x + 1 + 2y + 1 & \text{for some } x \text{ and } y \\ & & & \text{by the definition of odd} \\ & & & \text{and the inductive hypothesis} \\ & = & 2(x + y + 1) + 1 & \text{by algebra} \end{array}$$

And hence $\texttt{nodes}(T)$ is odd by definition of odd.

[Therefore, by the principle of structural induction, for any full binary tree T, $\texttt{nodes}(T)$ id odd.] □

For any full binary tree T, $\mathtt{nodes}(T) = 2 * \mathtt{internals}(T) + 1$.

Proof. *By induction on the structure of T.*

Base case. *Suppose T is a leaf. By definition of* internals*,* $\mathtt{internals}(T) = 0$*. Moreover, by definition of* nodes*,* $\mathtt{nodes}(T) = 1 = 2 \cdot 0 + 1 = 2 \cdot \mathtt{internals}(T) + 1$*.*

Inductive case. *Suppose T is an internal node with children T_1 and T_2 such that* $\mathtt{nodes}(T_1) = 2 \cdot \mathtt{internals}(T_1) + 1$ *and similarly for T_2.*

Then,

$$
\begin{aligned}
\mathtt{nodes}(T) &= 1 + \mathtt{nodes}(T_1) + \mathtt{nodes}(T_2) && \textit{by definition of nodes} \\
&= 1 + 2 \cdot \mathtt{internals}(T_1) + 1 + 2 \cdot \mathtt{internals}(T_2) + 1 && \textit{by the inductive hypothesis} \\
&= 2(1 + \mathtt{internals}(T_1) + \mathtt{internals}(T_2)) + 1 && \textit{by algebra} \\
&= 2\mathtt{internals}(T) + 1 && \textit{by definition of internals}
\end{aligned}
$$

[Therefore, by the principle of structural induction, for any full binary tree T, $\mathtt{nodes}(T) = 2 * \mathtt{internals}(T) + 1$*.]* □

For any full binary tree T, $\texttt{height}(T) \leq \texttt{links}(T)$.

Proof. *By induction on the structure of T.*

Base case. *Suppose T is a leaf. By definition of height and links,* $\texttt{height}(T) = 0 \leq 0 = \texttt{links}(T)$.

Inductive case. *Suppose T is an internal node with children T_1 and T_2 such that* $\texttt{height}(T_1) \leq \texttt{links}(T_1)$ *and similarly for T_2.*

[By definition of height and links, $\texttt{height}(T) = 1 + \max(\texttt{height}(T_1), \texttt{height}(T_2))$ *and* $\texttt{links}(T) = 2 + \texttt{links}(T_1) + \texttt{links}(T_2)$*.]*

Then

$$
\begin{aligned}
\texttt{height}(T) &= 1 + \max(\texttt{height}(T_1), \texttt{height}(T_2)) && \text{by definition of } \mathit{height} \\
&\leq 1 + \max(\texttt{links}(T_1), \texttt{links}(T_2)) && \text{by the inductive hypothesis} \\
&\leq 1 + \texttt{links}(T_1) + \texttt{links}(T_2) && \text{since the sum of nonnegatives is geq their max} \\
&< 2 + \texttt{links}(T_1) + \texttt{links}(T_2) && \text{since } 1 < 2
\end{aligned}
$$

[Therefore, by the principle of structural induction, for any full binary tree T, $\texttt{height}(T) \leq \texttt{links}(T)$*.]* □

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

```
fun factorial(0) = 1
  | factorial(n) = n * factorial(n-1);
```

Theorem 6.6. For all $n \in \mathbb{W}$, $\texttt{factorial}(n) = n!$

Proof. *By induction on n.*

Base case. *Suppose $n = 0$. By definition of* $\mathtt{factorial}$, $\texttt{factorial}(0) = 1 = 0!$, *by definition of* $!$. *Hence there exists an $N \geq 0$ such that* $\texttt{factorial}(N) = N!$.

Inductive case. *Suppose $N \geq 0$ such that* $\texttt{factorial}(N) = N!$, *and suppose* $n = N + 1$. *Then*

$$\begin{aligned}
\texttt{factorial}(n) &= n \cdot \texttt{factorial}(n-1) && \textit{by definition of factorial} \\
&= n \cdot \texttt{factorial}(N) && \textit{by algebra and substitution} \\
&= n \cdot N! && \textit{by the inductive hypothesis} \\
&= n! && \textit{by definition of !}
\end{aligned}$$

Therefore, by math induction, $\mathtt{factorial}$ *is correct for all* $n \in \mathbb{W}$. □

What does *correctness* mean for an algorithm?

The outcome/result must aways match the specification. For `arithSum`, the specification is

$$\texttt{arithSum}(N) = \sum_{k=1}^{N} k$$

To prove this, we need to reason about the *change of state* of the computation.

The *state* of the computation is represented by the values of the variables.

We can reason about a single line of code in terms of *preconditions* and *postconditions*.

Suppose the preconditions include $x = 5$.

```
y := x + 1
```

Then the postconditions include

- $y = 6$
- $x = 5$
- $x = y - 1$
- $G = 6.674 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg}\ \mathrm{s}^2}$

```
fun remainder(a, b) =
    let
```

Suppose $a, b \in \mathbb{Z}$

```
        val q = a div b;
```

$q = a$ div b by assignment. By the QRT (Thm 4.21) and the definition of division, $a = b \cdot q + R$ for some R, $0 \leq R < b$. Then by algebra, $q = \frac{a-R}{b}$.

```
        val p = q * b;
```

$p = q \cdot b$ by assignment, and $p = a - R$ by substitution and algebra.

```
        val r = a - p;
```

By assignment, $r = a - p$. By substitution and algrebra, $r = a - (a - R) = R$.

```
    in
        r
    end;
```

Since r is the value returned and is equal to the specified result R, this program returns the correct result. $\square$

For `arithSum`, N is the limit on the summation. Let n be the *number of iterations so far*. Our claim is

$$\text{After } n \text{ iterations, } \quad s = \sum_{k=1}^{n} k$$

Notice

- After 0 iterations, $s = 0$ and $\sum_{k=1}^{0} k = 0$. Our claim is true *before we start*.
- Each iteration changes the state, but maintains the fact above (or, so we claim).
- When we're done, that's N iterations, so $\sum_{k=1}^{n} k = \sum_{k=1}^{N} k$ (or, so we claim).

Refining the claim:

$$\forall\ n \in \mathbb{W}, \ \text{after } n \text{ iterations } s = \sum_{k=1}^{n} k \text{ and } i = n + 1$$

Theorem. `arithSum(N)` returns $\sum_{k=1}^{N} k$.

Lemma. $\forall\ n \in \mathbb{W}$, after n iterations, $s = \sum_{k=1}^{n} k$ and $i = n + 1$.

Proof (of lemma). By induction on the number of iterations, n.
Initialization. After 0 iterations, $s = 0 = \sum_{k=1}^{0} k$ by assignment, arithmetic, and definition of summation. $i = 1 = 0 + 1$, by assignment and arithmetic.
Maintenance. Suppose after $n \geq 0$ iterations, $s = \sum_{k=1}^{n} k$ and $i = n + 1$. Let s_{old} be s after n iterations and s_{new} be s after $n + 1$ iterations. Similarly define i_{old} and i_{new}. Then

$$
\begin{array}{rcll}
s_{\text{new}} & = & s_{\text{old}} + i_{\text{old}} & \text{by assignment} \\
 & = & (\sum_{k=1}^{n} k) + n + 1 & \text{by the inductive hypothesis} \\
 & = & \sum_{k=1}^{n+1} k & \text{by the definition of summation} \\
i_{\text{new}} & = & i_{\text{old}} + 1 & \text{by assignment} \\
 & = & n + 1 + 1 & \text{by the inductive hypothesis} \\
 & = & (n + 1) + 1 & \text{by associativity}
\end{array}
$$

Therefore the invariant holds. $\square$

Theorem. `arithSum(N)` returns $\sum_{k=1}^{N} k$.

Lemma. $\forall\ n \in \mathbb{W}$, after n iterations, $s = \sum_{k=1}^{n} k$ and $i = n + 1$.

Proof (of theorem). Suppose $N \in \mathbb{W}$ is the input to `arithSum`.

Termination. The lemma tells us that after N iterations, $i = N + 1 \not\leq N$, so the guard fails and the loop terminates.

At loop exit, $s = \sum_{k=1}^{N} k$, which is return.

Therefore the program `arithSum` is correct. □

Principles of using loop invariants to prove correctness

- A *loop invariant* is a proposition that is true before and after each iteration of a loop, including before the entire loop starts and after it terminates. A *useful* loop invariant captures an aspect of the progress of the loop's work.

- The steps in a loop invariant proof, to prove and apply something in the form, "$\forall n \in \mathbb{W}$, after n iterations, ..."
  - **Initialization.** Prove that the property is true before the loop starts, that is, after 0 iterations. This is the base case in the inductive proof.
  - **Maintenance.** Prove that *if* the property is true before an iteration, *then* it is true after that iteration. This is the inductive case of the inductive proof.
  - **Termination.** Prove that the loop *will terminate*, and then apply the loop invariant to deduce a postcondition for the entire loop.

```
fun aaa(m) =
  let
    val x = ref 0;
    val i = ref 0;
  in
    (while !i < m do
      (x := !x + 2 * !i;
       i := !i + 1);
     !x)
  end;
```

After n iterations, x is even.

Proof. By induction on the number of iterations.
Initialization. Before the loop starts, $x = 0$ by assignment. Moreover, $x = 2 \cdot 0$, so x is even by definition.
Maintenance. Suppose that after n iterations x is even, for some $n \geq 0$. Let x_{old} and x_{new} be x after n and $n+1$ iterations, respectively.
$x_{\text{old}} = 2j$ for some $j \in \mathbb{Z}$ by the inductive hypothesis and definition of even. Then

$$\begin{aligned} x_{\text{new}} &= x_{\text{old}} + 2i && \text{by assignment} \\ &= 2j + 2i && \text{by substitution} \\ &= 2(j+i) && \text{by algebra} \end{aligned}$$

Hence x_{new} is even by definition.
Therefore, by the principle of mathematical induction, that x is even is a loop invariant. □

```
fun pow(x, y) =
    let
        val a = ref 1;
        val i = ref y;
    in
       (while  !i > 0 do
            (i := !i - 1;
             a := !a * x);
        !a)
end;
```

After n iterations, $a = x^n$ and $i = y - n$.

Proof. By induction on the number of iterations.

Initialization. Suppose $n = 0$, that is, the conditions before the loop starts. Then $a = 1$ by assignment, and hence $a = x^0 = x^n$ by algebra. Similarly, $i = y$ by assignment, and hence $i = y - 0 = y - n$ by algebra.

Maintenance. Suppose that $a = x^n$ and $i = y - n$ after n iterations for some $n \geq 0$. Let a_{old}, a_{new}, i_{old}, and i_{new} be defined in the usual way. Then

$$
\begin{array}{rcll}
i_{\text{new}} & = & i_{\text{old}} - 1 & \text{by assignment} \\
 & = & y - n - 1 & \text{by the inductive hypothesis} \\
 & = & y - (n+1) & \text{by algebra} \\
a_{\text{new}} & = & a_{\text{old}} \cdot x & \text{by assignment} \\
 & = & x^n \cdot x & \text{by the inductive hypothesis} \\
 & = & x^{n+1} & \text{by algebra}
\end{array}
$$

Therefore, by the principle of mathematical induction, $a = x^n$ and $i = y - n$, where n is the number of iterations completed, is a loop invariant. □

After n iterations, $x + y = m$.

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

After n iterations, $x + y = m$.

Proof. By induction on the number of iterations.

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

After n iterations, $x + y = m$.

Proof. By induction on the number of iterations.

Initialization. Before the loop starts, $x = m$ and $y = 0$ by assignment. Hence $x + y = m$ by algebra.

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

After n iterations, $x + y = m$.

Proof. By induction on the number of iterations.
Initialization. Before the loop starts, $x = m$ and $y = 0$ by assignment. Hence $x + y = m$ by algebra.
Maintenance Suppose $x + y = m$ after n iterations for some $n \geq 0$. Let x_{old}, x_{new}, y_{old}, and y_{new} be defined in the usual way. Then

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

After n iterations, $x + y = m$.

Proof. By induction on the number of iterations.
Initialization. Before the loop starts, $x = m$ and $y = 0$ by assignment. Hence $x + y = m$ by algebra.
Maintenance Suppose $x + y = m$ after n iterations for some $n \geq 0$. Let x_{old}, x_{new}, y_{old}, and y_{new} be defined in the usual way. Then

$$\begin{aligned} x_{new} &= x_{old} - i && \text{by assignment} \\ y_{new} &= y_{old} + i && \text{by assignment} \\ x_{new} + y_{new} &= x_{old} - i + y_{old} + i && \text{by substitution} \\ &= x_{old} + y_{old} && \text{by algebra} \\ &= m && \text{by the inductive hypothesis} \end{aligned}$$

```
fun xxx(m) =
  let
    val x = ref m;
    val y = ref 0;
    val i = ref 1;
  in
   (while !i < m div 2 do
     (x := !x - i;
      y := !y + i;
      i := !i * 2);
    !x - !y)
  end;
```

After n iterations, $x + y = m$.

Proof. By induction on the number of iterations.
Initialization. Before the loop starts, $x = m$ and $y = 0$ by assignment. Hence $x + y = m$ by algebra.
Maintenance Suppose $x + y = m$ after n iterations for some $n \geq 0$. Let x_{old}, x_{new}, y_{old}, and y_{new} be defined in the usual way. Then

$$\begin{aligned}
x_{new} &= x_{old} - i && \text{by assignment} \\
y_{new} &= y_{old} + i && \text{by assignment} \\
x_{new} + y_{new} &= x_{old} - i + y_{old} + i && \text{by substitution} \\
&= x_{old} + y_{old} && \text{by algebra} \\
&= m && \text{by the inductive hypothesis}
\end{aligned}$$

Therefore, by the principle of mathematical induction, $x + y = m$ is a loop invariant. □

Reminder: Ex 6.10.(2-5) for next time.
Also (very important):

- Read 7 intro and 7.1 *carefully*
- Read 7.2
- Skim 7.3
- Take quiz