Chapter 6, Hash tables:

- General introduction; separate chaining (week-before Friday)
- Open addressing (last week Wednesday)
- Hash functions (last week Friday)
- Perfect hashing (Monday)
- Hash table performance (**Today**)

Today:

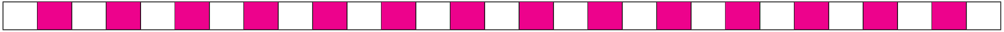
- Finishing perfect hashing
- Elements of hashtable performance
- Separate chaining performance
- Open addressing performance

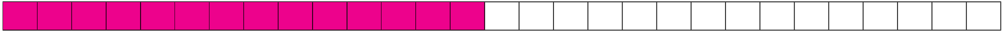
	Find	Insert	Delete
Unsorted array	$\Theta(n)$	$\Theta(1)$ $[\Theta(n)]$	$\Theta(n)$
Sorted array	$\Theta(\lg n)$	$\Theta(n)$	$\Theta(n)$
Linked list	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
Balanced BST	$\Theta(\lg n)$	$\Theta(1)$ $[\Theta(\lg n)]$	$\Theta(1)$ $[\Theta(\lg n)]$
What we want	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Find	Search the data structure for a given key
Insert	Add a new key to the data structure
Delete	Get rid of a key and fix up the data structure

`containsKey()`	Find
`get()`	Find
`put()`	Find + insert
`remove()`	Find + delete

$$
\left.\begin{array}{rll}
 & O(1) & c_0 \\
 & O(1) & c_0 \\
 & O(1) & c_0 \\
 & \vdots & \\
 & O(1) & c_0 \\
\text{rehash} \longrightarrow & O(n) & c_1 + c_2 n \\
 & O(1) & c_0 \\
 & \vdots & \\
 & O(1) & c_0
\end{array}\right\}
\begin{array}{rcl}
T(n) & = & (n-1)c_0 + c_1 + c_2 n \\
 & = & (c_0 + c_2)n + c_1 - c_0 \\
 & = & \Theta(n)
\end{array}
$$





$$\frac{(n+1) + n + (n-1) + \cdots + 3 + 2 + \overbrace{1 + \cdots + 1}^{m-n}}{m}$$

$$= \frac{m + n + (n-1) + \cdots + 2 + 1}{m}$$

the initial m accounting for the last probe in each case

$$= \frac{m}{m} + \frac{(n+1) \cdot \frac{n}{2}}{m}$$

as an arithmetic series

$$\approx 1 + \frac{(n+1) \cdot \frac{n}{2}}{2 \cdot n}$$

since m is about $2 \cdot n$

$$= 1 + \frac{n+1}{4}$$

by cancellation

$$\frac{[(s_0 + 1) + s_0 + (s_0 - 1) + \cdots + 2] + \cdots + 1 + \cdots 1}{m} = 1 + \frac{\sum_{i=0}^{\gamma-1} \sum_{j=1}^{s_i} j}{m}$$

What is the probability that a miss k requires at least i probes?

Conditional probability

$P(X \mid Y)$: What is the probability of event X in light of event Y?

$$P(X \wedge Y) = P(X) \cdot P(X \mid Y)$$

$$P(X_0 \wedge X_1 \wedge \cdots \wedge X_{N-1}) = P(X_0) \cdot P(X_1 \mid X_0) \cdot P(X_1 \mid X_0 \wedge X_1) \cdots P(X_{N-1} \mid X_0 \wedge \cdots \wedge X_{N-2})$$

$$P(\,T[h(k)+1] \neq \texttt{null} \mid T[h(k)] \neq \texttt{null}) = \frac{n-1}{m-1}$$

The probability that a miss requires at least i probes:

$$\frac{n}{m} \cdot \frac{n-1}{m-1} \cdots \frac{n-i+2}{m-i+2}$$

$$\leq \left(\frac{n}{m}\right)^{i-1} \qquad \text{since } n < m$$

$$\leq \alpha^{i-1} \qquad \text{by substitution}$$

$$\sum_{i=1}^{m} i \cdot P\begin{pmatrix}\text{it takes}\\ i \text{ probes}\end{pmatrix} = \sum_{i=1}^{m} i \cdot \left(P\begin{pmatrix}\text{it takes}\\ \text{at least } i\\ \text{probes}\end{pmatrix} - P\begin{pmatrix}\text{it takes at}\\ \text{least } i+1\\ \text{probes}\end{pmatrix}\right)$$

$$= \sum_{i=1}^{m} P\begin{pmatrix}\text{it takes}\\ \text{at least } i\\ \text{probes}\end{pmatrix} \quad \text{by telescoping}$$

$$\leq \sum_{i=1}^{m} \alpha^{i-1} \quad \text{by the previous result}$$

$$\leq \sum_{i=1}^{\infty} \alpha^{i-1} \quad \text{since } m < \infty$$

$$= \sum_{i=0}^{\infty} \alpha^{i} \quad \text{by a change of variable}$$

$$= \frac{1}{1-\alpha} \quad \text{by geometric series}$$

Is the following assumption true for linear probing?

$$P(T[h(k)+1] \neq \texttt{null} \mid T[h(k)] \neq \texttt{null}) = \frac{n-1}{m-1}$$

In general, is the following assumption true for a probing strategy?

$$P(T[\sigma(k,1)] \neq \texttt{null} \mid T[\sigma(k,0)] \neq \texttt{null}) = \frac{n-1}{m-1}$$

What is the difference between

Each array index is equally likely to be the hash of a given key.	vs	Each array position is equally likely to be occupied.

Linear probing is biased towards clustering:

x	Number of buckets with exactly x previous buckets filled	Number of filled buckets with exactly x previous buckets filled	Probability that a bucket is filled if exactly x previous buckets are filled.
0	97	48	.495
1	48	22	.458
2	22	12	.545
3	12	7	.583
4	7	4	.571
5	4	3	.75
6	3	2	.667
7	2	2	1
8	2	0	0

Expected number of probes for a miss in a hashtable using linear probing (from Knuth):

$$\frac{1}{2} \cdot \left(1 + \frac{1}{(1-\alpha)^2}\right)$$

After n calls to `put()` with unique keys, no removals, consider **average chain length** over all keys (low is good), **percent of keys that are in their ideal location** (high is good), and **length of the longest chain** (low is good)

	n	Linear probing			Quadratic probing			Double hashing		
Surnames	1000	2.092	64.7%	31	1.421	75.8%	9	2.327	65.2%	31
Mountains	1360	1.568	73.8%	17	1.729	65.8%	11	1.770	73.4%	16
Mountains (height)	1360	1.932	75.1%	99	1.882	68.9%	18	1.830	72.4%	13
Chemicals	663	1.517	75.0%	16	1.729	65.5%	10	1.701	75.5%	9
Chemicals (symbol)	663	1.885	71.0%	20	1.837	66.4%	13	1.798	72.7%	12
Books	718	1.419	76.7%	8	1.659	70.0%	11	1.656	75.8%	8
Books (ISBN)	718	1.542	74.4%	21	1.670	67.8%	15	1.724	74.5%	10
Random strings	5000	1.544	77.6%	49	1.735	69.9%	37	1.598	78.1%	13
Random strings	5000	1.531	77.1%	35	1.729	69.8%	28	1.593	77.9%	12
Random strings	5000	1.643	77.5%	76	1.754	68.6%	29	1.590	78.1%	13

Coming up:

Do **Open Addressing Hashtable** *project (due this past Monday, Apr 15)*
Do **Perfect Hashing** *project (due Monday, Apr 22)*

Due **Today, Wed, Apr 17 (end of day)**
Re-read end of Section 7.3
Take quiz

Due **Fri, Apr 19** *(end of day)*
Read Section 8.1
Do Exercises 8.(4 & 5)
Take the last quiz

Due **Mon, Apr 22** *(end of day)*
Read Section 8.2
(No quiz or practice problems)