Chapter 5 roadmap:

- Introduction to relations (spring-break eve)
- Properties of relations (**Today** and Wednesday)
- Closures (Friday)
- Partial order relations (next week Monday)

"Today" (Monday and Wednesday):

- Review of definitions from last time
- Revisit proofs from last time
- Hints on homework problems
- Properties of relations
  - Reflexivity
  - Symmetry
  - Transitivity
- Proofs
- More proofs

Coming up:

Due **Wednesday, Mar 19**:
Do Exercises 5.1.5 and 5.2.(7,8,10,11, 12, 13, 14, 15)
See Canvas for hints/explanations.
Read Section 5.3
Take quiz on Section 5.3

Note that Section 5.3 will take up two days (Monday and Wednesday). There will be no homework assignment due Monday. The homework for Section 5.3 will be due Friday.

Due **Friday**:
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A **relation** from one set to another	R	set of pairs	subset of $X \times Y$ $R \subseteq X \times Y$	isEnrolledIn, isTaughtBy
A **relation** on a set	R	set of pairs	subset of $X \times X$ $R \subseteq X \times X$	eats, divides
The **image** of an element under a relation	$\mathcal{I}_R(a)$	set	set of things that a is related to $\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$	classes Bob is enrolled in, numbers that 4 divides
The **image** of a set under a relation	$\mathcal{I}_R(A)$	set	set of things that things in A are related to $\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$	classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide
The **inverse** of a relation	R^{-1}	relation	the arrows/pairs of R reversed $R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$	hasOnRoster, teaches, isEatenBy, isDivisibleBy
The **composition** of two relations	$S \circ R$	relation	two hops combined to one hop (Assume $S \subseteq Y \times Z$) $S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y$ $\mid (a, b) \in R \wedge (b, c) \in S\}$	hasAsProfessor, eatsSomethingThatEats
The **identity** relation on a set	i_X	relation	everything is related only to itself $i_X = \{(x, x) \mid x \in X\}$	$=$

Theorem 5.1 If $a, b \in \mathbb{N}$ and $a|b$, then $\mathcal{I}_|(b) \subseteq \mathcal{I}_|(a)$.

Proof. *Suppose $a, b \in \mathbb{N}$ and $a|b$. By definition of divides, there exists $i \in \mathbb{N}$ such that $a \cdot i = b$.*

Suppose further that $c \in \mathcal{I}_|(b)$. By definition of image, $b|c$. By definition of divides, there exists $j \in \mathbb{N}$ such that $b \cdot j = c$.

By substitution, $a \cdot i \cdot j = c$, and so $a|c$ by definition of divides. By definition of image, $c \in \mathcal{I}_|(a)$, and by definition of subset, $\mathcal{I}_|(b) \subseteq \mathcal{I}_|(a)$. $\square$

Theorem 5.2 If R is a relation on a set A, $a \in A$, and $\mathcal{I}_R(a) \neq \emptyset$, then $a \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(a))$.

Proof. *Suppose R is a relation on A, $a \in A$, and $\mathcal{I}_R(a) \neq \emptyset$.*

Let $b \in \mathcal{I}_R(a)$. By definition of image, $(a, b) \in R$. By definition of inverse, $(b, a) \in R^{-1}$. By definition of image (extended for sets), $a \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(a))$. $\square$

Ex 5.2.7. Prove that if R is a relation on a set A and $(a, b) \in R$, then $\mathcal{I}_R(b) \subseteq \mathcal{I}_{R \circ R}(a)$.

Ex 5.2.8. Suppose R is a relation from a set X to a set Y and $A \subseteq X$. Is the following true?
$\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$.
Prove or give a counterexample.

Ex 5.2.8. Suppose R is a relation from a set X to a set Y and $A \subseteq X$. Is the following true?
$\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$.
Prove or give a counterexample.

Attempted proof. *Suppose* $x \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(A))$. *[We want* $x \in A$*.]*

By definition of image, there exists $y \in \mathcal{I}_R(A)$ *such that* $(y, x) \in R^{-1}$.

[From $y \in \mathcal{I}_R(A)$*]*
By definition of image, there exists $a \in A$ *such that* $(a, y) \in R$.

[From $(y, x) \in R^{-1}$*]*
By definition of relation inverse, $(x, y) \in R$

[We know $a \in A$*, and both* $(a, y) \in R$ *and* $(x, y) \in R$. *Could it be that* $a = x$*? Doesn't seem to be a way to prove that... I seem stuck]*

Counterexample. *Let* $X = \{x, a\}$, $A = \{a\}$, *and* $Y = \{y\}$.
Let $R = \{(x, y), (a, y)\}$.
Then $R^{-1} = \{(y, x), (y, a)\}$, $\mathcal{I}_R(A) = \{y\}$, *and* $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) = \{x, a\}$
In this example, $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \not\subseteq A$.

Ex 5.2.9. Prove that if R is a relation from A to B, then $i_B \circ R = R$.

Ex 5.2.9. Prove that if R is a relation from A to B, then $i_B \circ R = R$.

Proof. *First suppose* $(x, y) \in i_B \circ R$*. By definition of composition, there exists* $b \in B$ *such that* $(x, b) \in R$ *and* $(b, y) \in i_B$*.*

By definition of the identity relation, $b = y$*. By substitution,* $(x, y) \in R$*. Hence* $i_B \circ R \subseteq R$ *by definition of subset.*

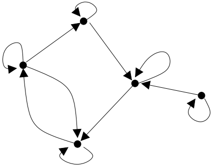
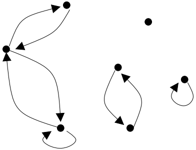
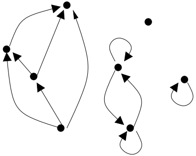
Next suppose $(x, y) \in R$*. By how* R *is defined, we know* $x \in A$ *and* $y \in B$*.*

By definition of the identity relation, $(y, y) \in i_B$*. By definition of composition,* $(x, y) \in i_B \circ R$*. Hence* $R \subseteq i_B \circ R$*.*

Therefore, by definition of set equality, $i_B \circ R = R$*.* □

Ex 5.2.10. $(R^{-1})^{-1} = R$.

Ex 5.2.11. If R is a relation from A to B, is $R^{-1} \circ R = i_A$? Prove or give a counterexample.

	Reflexivity	**Symmetry**	**Transitivity**
Informal	Everything is related to itself	All pairs are mutual	Anything reachable by two hops is reachable by one hop
Formal	$\forall\ x \in X,\ (x,x) \in R$	$\forall\ x,y \in X,\ (x,y) \in R \rightarrow (y,x) \in R$ OR $\forall (x,y) \in R,\ (y,x) \in R$	$\forall x,y,z \in X,$ $(x,y),(y,z) \in R \rightarrow (x,z) \in R$ OR $\forall (x,y),(y,z) \in R, (x,z) \in R$
Visual			
Examples	$\subseteq$, $\leq$, $\geq$, $\equiv$, i, isAquaintedWith, waterVerticallyAligned	$\equiv$, isOppositeOf, isOnSameRiver, isAquaintedWith	$<$, $\leq$, $>$, $\geq$, $\subseteq$, isTallerThan, isAncestorOf, isWestOf

	Reflexivity	**Symmetry**	**Transitivity**
Formal	$\forall\ x \in X,\ (x, x) \in R$	$\forall\ x, y \in X$, $(x, y) \in R \to (y, x) \in R$ OR $\forall\ (x, y) \in R,\ (y, x) \in R$	$\forall\ x, y, z \in X$, $(x, y), (y, z) \in R \to (x, z) \in R$ OR $\forall\ (x, y), (y, z) \in R, (x, z) \in R$
Analytical use	Suppose R is reflexive and $a \in X$. Then $(a, a) \in R$.	Suppose R is symmetric $[a, b \in X]$ and $(a, b) \in R$. Then $(b, a) \in R$	Suppose R is transitive $[a, b, c \in X]$ and $(a, b), (b, c) \in R$. Then $(a, c) \in R$.
Synthetic use	Suppose $a \in X$. ... $(a, a) \in R$. Hence R is reflexive.	Suppose $(a, b) \in R$. ... $(b, a) \in R$. Hence R is symmetric.	Suppose $(a, b), (b, c) \in R$. ... $(a, c) \in R$. Hence R is transitive.

Theorem 5.5. $\mid$ (divides) is reflexive.

Exercise 5.3.1. $\mid$ (divides) is not symmetric.

Theorem 5.6. $R \cap R^{-1}$ is symmetric.

Theorem 5.7. $\mid$ is transitive.

Exercise 5.3.19. $R^{-1} \circ R$ is reflexive. *(False)*

Exercise 5.3.20. If R and S are both reflexive, then $R \cap S$ is reflexive.

Exercise 5.3.22. If R and S are both symmetric, then $(S \circ R) \cup (R \circ S)$ is symmetric.

Based on Exercise 5.3.32. If R is transitive, then $R \circ R \subseteq R$.

Exercise 5.3.26. If R is transitive, $\mathcal{I}_R(\mathcal{I}_R(A)) \subseteq \mathcal{I}_R(A)$.

Exercise 5.3.31. If R is reflexive and

(for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(c, a) \in R$),

then R is an equivalence relation.

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