Chapter 6 in context:

- Chapter 5 Relations: Builds on proofs about sets
- Chapter 6 Function: Builds on proofs about relations
- Chapter 7 Self Reference: Focuses on recursive thinking

Chapter 6 outline:

- Introduction, function equality, and dictionaries (**Today**)
- Image and inverse images (Friday)
- Function properties and composition (next week Monday)
- Map, reduce, filter (next week Wednesday)
- Cardinality (next week Friday)
- Countability (week-after Monday, Apr 7)
- Review (week-after Wednesday, Apr 9)
- Test 3, on Ch 5 & 6 (week-after Friday, Apr 11)

Ex 5.2.8. Suppose R is a relation from a set X to a set Y and $A \subseteq X$. Is the following true?
$\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \subseteq A$.
Prove or give a counterexample.

Attempted proof. *Suppose* $x \in \mathcal{I}_{R^{-1}}(\mathcal{I}_R(A))$. *[We want* $x \in A$.*]*

By definition of image, there exists $y \in \mathcal{I}_R(A)$ *such that* $(y, x) \in R^{-1}$.

[From $y \in \mathcal{I}_R(A)$*]*
By definition of image, there exists $a \in A$ *such that* $(a, y) \in R$.

[From $(y, x) \in R^{-1}$*]*
By definition of relation inverse, $(x, y) \in R$

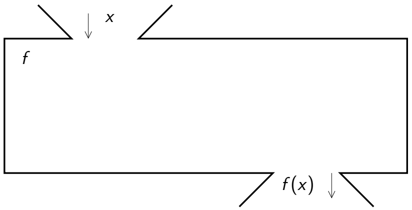
[We know $a \in A$, *and both* $(a, y) \in R$ *and* $(x, y) \in R$. *Could it be that* $a = x$? *Doesn't seem to be a way to prove that... I seem stuck]*

Counterexample. *Let* $X = \{x, a\}$, $A = \{a\}$, *and* $Y = \{y\}$.
Let $R = \{(x, y), (a, y)\}$.
Then $R^{-1} = \{(y, x), (y, a)\}$, $\mathcal{I}_R(A) = \{y\}$, *and* $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) = \{x, a\}$
In this example, $\mathcal{I}_{R^{-1}}(\mathcal{I}_R(A)) \not\subseteq A$.

What about $A \subseteq \mathcal{I}_{R-1}(\mathcal{I}(A))$?

A function is. . .

- a parameterized expression.
- a named piece of code that can be invoked many times in different contexts.
- an extension to the programming language.
- an abstract machine.
- a *value*.



Cross out the term/concept that was **not** used in the reading for today as a way to think about functions

A kind of machine

A topological sort

A mapping between two collections

A kind of relation

For the function $f : X \to Y$, X is the ____________________ and Y is the

____________________.

function

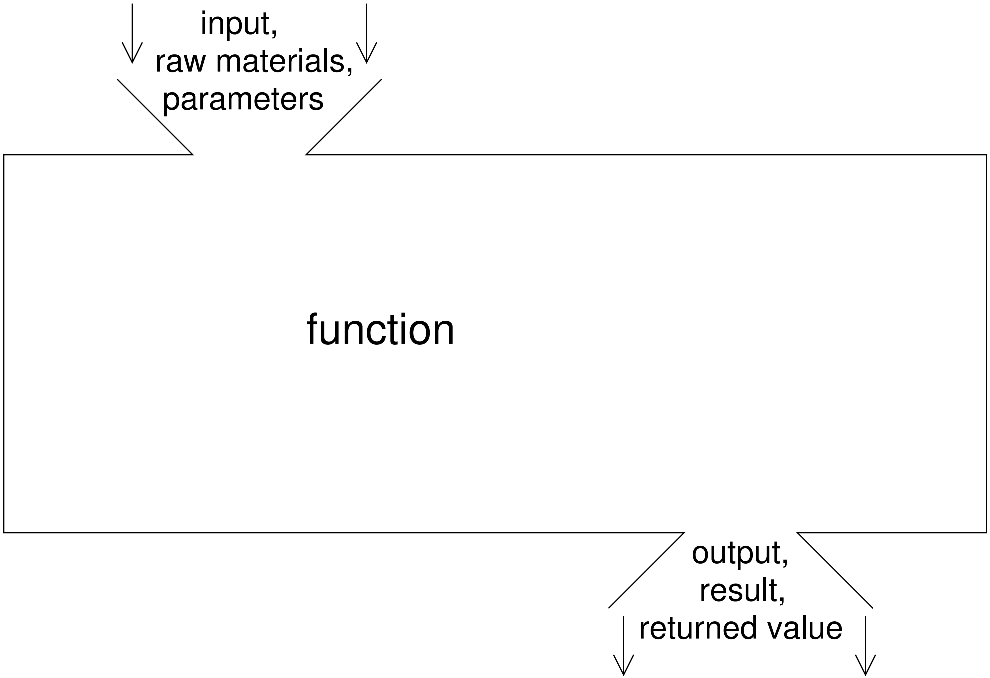
constant

domain

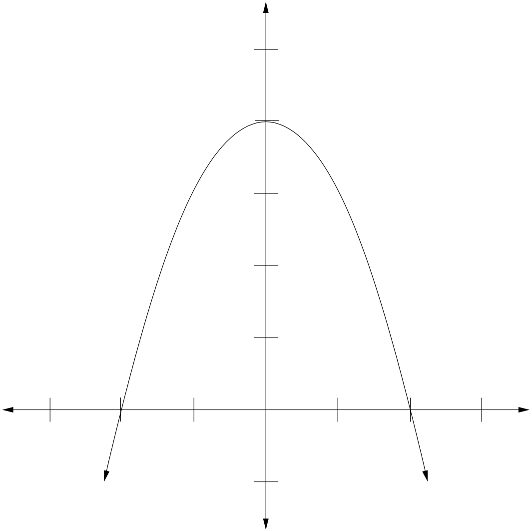
codomain

first-class value

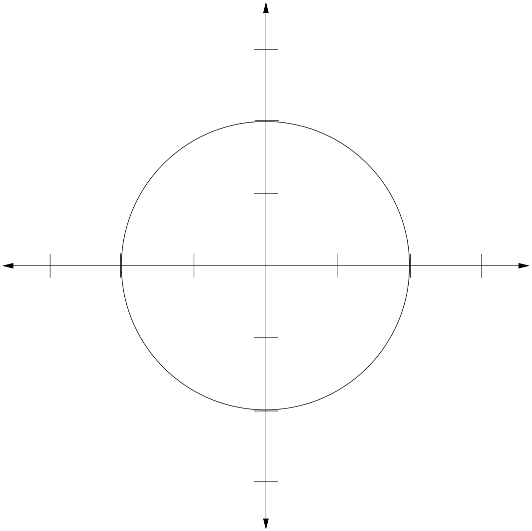
relation


input,
raw materials,
parameters
function
output,
result,
returned value

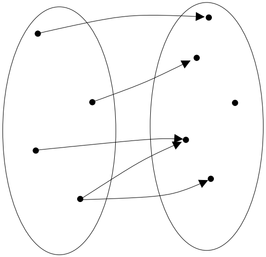
Alice	x3498
Bob	x4472
Carol	x5392
Dave	x9955
Eve	x2533
Fred	x9448
Georgia	x3684
Herb	x8401



$y = 4 - x^2$

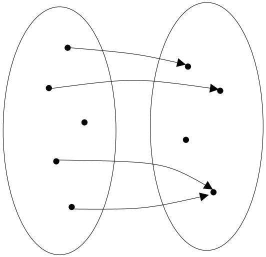


$y^2 = 4 - x^2$



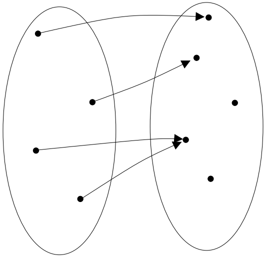
Not a function.

(There's a domain element that is related to two things.)



Not a function.

(There's a domain element that is not related to anything.)



A function.

(It's OK that two domain elements are related to the same thing and one codomain element has nothing related to it.)

Definition of function

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal: $f \subseteq X \times Y$ is a *function* if

$$\begin{aligned} \forall\, x \in X, \quad & \exists\, y \in Y \mid (x, y) \in f && \textbf{existence of } y \\ \wedge \quad & \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2 && \textbf{uniqueness of } y \end{aligned}$$

Change of notation

Informal: A *function* is a relation in which everything in the first set is related to *exactly one thing* in the second set.

Formal (relation notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \qquad \exists\, y \in Y \mid (x, y) \in f$$
existence of y

$$\wedge \quad \forall\, y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \to y_1 = y_2$$
uniqueness of y

Formal (function notation): $f \subseteq X \times Y$ is a *function* if

$$\forall\, x \in X, \qquad \exists\, y \in Y \mid f(x) = y$$
existence of y

$$\wedge \quad \forall\, y_1, y_2 \in Y, (f(x) = y_1 \wedge f(x) = y_2) \to y_1 = y_2$$
uniqueness of y

We call X the *domain* and Y the *codomain* of f.

Definition of function equality. Let $f, g : X \to Y$

Old definition: functions are sets.

$$f = g \text{ if } \forall\, f \subseteq g \;\wedge g \subseteq f$$

New definition: based on function notation.

$$f = g \;\text{ if }\; \forall\, x \in X, f(x) = g(x)$$

Function equality: $f = g$ if $\forall\ x \in X, f(x) = g(x)$

Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x \cdot (x - 1) - 6$ and $g(x) = (x - 3)(x + 2)$.

Prove $f = g$.

The old and new definitions of function equality are equivalent.

Ex 7.2.1. $(\forall\, x \in X, f(x) = g(x)) \quad \text{iff} \quad (f \subseteq g \wedge g \subseteq f)$.

The old and new definitions of function equality are equivalent.

Ex 7.2.1. $(\forall\, x \in X, f(x) = g(x))$ iff $(f \subseteq g \wedge g \subseteq f)$.

Proof. First, suppose $\forall\, x \in X, f(x) = g(x)$, that is, $f = g$ by definition of function equality. Further suppose $(x, y) \in f$. By function notation, $f(x) = y$. By supposition and substitution, $g(x) = y$. By relation notation, $(x, y) \in g$. Finally, $f \subseteq g$ by definition of subset.

Similarly $g \subseteq f$, and therefore $f = g$ by definition of set equality.

Conversely, suppose $f \subseteq g \wedge g \subseteq f$, that is, $f = g$ by definition of set equality. Further suppose $x \in X$.

Let $y = f(x)$. Note that this $y \in Y$ must exist by definition of function. By relation notation, $(x, y) \in f$.

By definition of subset [or set equality], $(x, y) \in g$. In function notation, that is $g(x) = y$, and so $f(x) = g(x)$ by substitution. Therefore $f = g$ by definition of function equality. $\square$

For next time:

Do Exercises 6.1.(2,3,7,8,9,10,11).

Exercises 2 and 3 are function-equality proofs. The other exercises are programming problems.

Read Section 6.2.

Take quiz